Mathematics Department
Prof. Dr. J. Behrens

SuSe 2023

Exam Differential Equations II

05. September 2023

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Task no.	Points	Evaluater
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Exercise 1: [5 points]

Compute the solution to the following initial value problem for $u(x,t)$:

$$u_t - 4t^3 u_x = e^{-t}, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$
$$u(x,0) = 1 + \sin(2x) \qquad x \in \mathbb{R}.$$

Solution: Using the method of characteristics we obtain:

$\frac{dx}{dt} = -4t^3 \implies dx = -4t^3 dt \implies x = -t^4 + C_1$ **[1 point]**

$\frac{du}{dt} = e^{-t} \implies du = e^{-t} dt \implies u = -e^{-t} + C_2\,.$ **[1 point]**

With $C_1 = x + t^4$ and $C_2 = u + e^{-t}$

we use the ansatz

$C_2 = f(C_1)$

and obtain

$u + e^{-t} = f(x + t^4)$

and thus the general solution is: $u(x,t) = f(x + t^4) - e^{-t}$. **[1 point]**

The initial condition requires:

$u(x,0) = f(x + 2 \cdot 0^2) - e^{-0} = f(x) - 1 \stackrel{!}{=} 1 + \sin(2x)$.

So $f(x) = 2 + \sin(2x)$ and

$u(x,t) = 2 + \sin(2x + 2t^4) - e^{-t}$. **[2 points]**

Exercise 2: [4= 2+2 points]

a) The functions

$$u_1(x,t) = \begin{cases} 0 & x \le 1+t \\ 2 & x > 1+t \end{cases} \qquad \text{and} \qquad u_2(x,t) = \begin{cases} 0 & x \le 1 \\ \frac{x-1}{t} & 1 < x < 2t+1 \\ 2 & 2t+1 \le x \end{cases}$$

are both weak solutions to the following initial value problem

$$u_t + \left(\frac{u^2}{2}\right)_x = u_t + u \cdot u_x = 0, \qquad u(x,0) = \begin{cases} 0 & x \le 1, \\ 2 & x > 1 \end{cases} \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+.$$

Determine and justify which of the two weak solutions is the entropy solution.

b) Given is the initial value problem

$$v_t + \left(\frac{v^2}{2}\right)_x = v_t + v \cdot v_x = 0, \qquad v(x,0) = \begin{cases} 2 & x \le 0, \\ -2 & x > 0. \end{cases}$$

Determine and justify which of the following functions v_1 or v_2

$$v_1(x,t) = \begin{cases} 2 & x \le 0 \\ -2 & x > 0 \end{cases} \qquad v_2(x,t) = \begin{cases} 2 & x \le 2t \\ -2 & x > 2t \end{cases}$$

is the entropy solution to the initial value problem.

Solution:

a) Here the initial data is increasing and f is convex. It holds
$f'(u_l) = u_l = 0 < 2 = u_r = f'(u_r)$.
The shock front of an entropy condition would have to satisfy the entropy condition:
$f'(u_l) = u_l > \dot{s}(t) > u_r = f'(u_r)$.
u_1 is not an entropy solution. u_2 is an entropy solution (standard rarefaction wave).

b) The initial data is decreasing, so a shock wave is introduced. The jump condition reads as
$$\dot{s}(t) = \frac{v_l + v_r}{2} = \frac{2-2}{2} = 0.$$
And thus v_1 is the entropy solution.

Exercise 3: [1+1+2 points]

Given the differential equation

$$4u_{xx} - 2u_{xt} + 4u_{tt} \; = \; 0 \quad \text{for } x \in \mathbb{R}, t > 0$$

a) rewrite the differential equation in matrix notation.

b) determine the type of the differential equation (elliptic, hyperbolic or parabolic).

c) transform the differential equation to a diagonal form $\alpha \cdot \tilde{u}_{\eta\eta} + \beta \cdot \tilde{u}_{\tau\tau} = 0$.

Solution:

$$4u_{xx} + 2 \cdot (-1)u_{xt} + 4u_{tt} \; = \; 0 \quad \text{for } x \in \mathbb{R}, t > 0\,.$$

a) With $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$ we have the matrix notation

$\nabla^T \cdot A \cdot \nabla\, u = 0\,.$ **(1 point)**

b) From $4 \cdot 4 - (-1)^2 > 0$ we have $\det(A) = 16 - 1 > 0$, so the differential equation is elliptic. **(1 point)**

c) Eigenvalues of A:

$(4 - \lambda)(4 - \lambda) - 1 = 0 \Longrightarrow (4 - \lambda)^2 = 1 \Longrightarrow 4 - \lambda = \pm 1.$

We obtain eigenvalues $\lambda_1 = 3\,, \lambda_2 = 5$ **(1 point)**

and hence the diagonal form is

$$3\tilde{u}_{\eta\eta} + 5\tilde{u}_{\tau\tau} = 0.$$ **(1 point)**

Exercise 4: [6+1 point]

a) Given is the initial boundary value problem

$$\begin{aligned} u_{tt} - 9u_{xx} &= \left(\frac{x}{2} - 1\right)\sin(t) && \text{for } x \in (0,2), t > 0, \\ u(x,0) &= \frac{x}{4} && \text{for } x \in [0,2], \\ u_t(x,0) &= 1 - \frac{x}{2} && \text{for } x \in [0,2], \\ u(0,t) &= \sin(t), \quad u(2,t) = \frac{1}{2} && \text{for } t > 0. \end{aligned}$$

Using a suitable homogenization of the boundary data, transform the problem into an initial boundary value problem with homogeneous boundary data for a function $v(x,t)$.

Provide the new initial boundary value problem (differential equation, initial conditions and boundary conditions).

b) Determine the solution to the initial boundary value problem for $v(x,t)$ from a).

Solution:

a) Homogenization:

$v(x,t) = u(x,t) - \left[\sin(t) + \frac{x}{2}(\frac{1}{2} - \sin(t))\right] = u(x,t) - \frac{x}{4} - \sin(t)(1 - \frac{x}{2})$.

or

$u(x,t) = v(x,t) + \frac{x}{4} + \sin(t)(1 - \frac{x}{2})$. **[1 point]**

Then it holds:

$u_t = v_t + \cos(t)(1 - \frac{x}{2})$, $u_{tt} = v_{tt} - \sin(t)(1 - \frac{x}{2})$, $v_{xx} = u_{xx}$ **[1 point]**

New differetial equation: $v_{tt} - sin(t)(1 - \frac{x}{2}) - 9v_{xx} = \left(\frac{x}{2} - 1\right)\sin(t) \Longleftrightarrow$

$\boxed{v_{tt} - 9v_{xx} = 0.}$ **[1 point]**

Initial data:

$v(x,0) = u(x,0) - \left[\sin(0) + \frac{x}{2}(\frac{1}{2} - \sin(0))\right] = \frac{x}{4} - \frac{x}{4} = 0.$

$\boxed{v(x,0) = 0}$ **[1 point]**

$v_t(x,0) = u_t(x,0) - \cos(0)(1 - \frac{x}{2}) = 1 - \frac{x}{2} - (1 - \frac{x}{2}) = 0.$

$\boxed{v_t(x,0) = 0}$ **[1 point]**

$v(0,t) = u(0,t) - \left[\sin(t) + \frac{0}{2}(\frac{1}{2} - \sin(t))\right] = \sin(t) - \sin(t) = 0.$

$v(2,t) = u(2,t) - \left[\sin(t) + \frac{2}{2}(\frac{1}{2} - \sin(t))\right] = \frac{1}{2} - \sin(t) - \frac{1}{2} + \sin(t) = 0.$

Boundary data: $\boxed{v(0,t) = v(2,t) = 0}$ **[1 point]**

b) Since all right hand sides of the initial boundary value problem for v are zeros, $v(x,t) \equiv 0$ is the solution to the problem. [**1 point**]