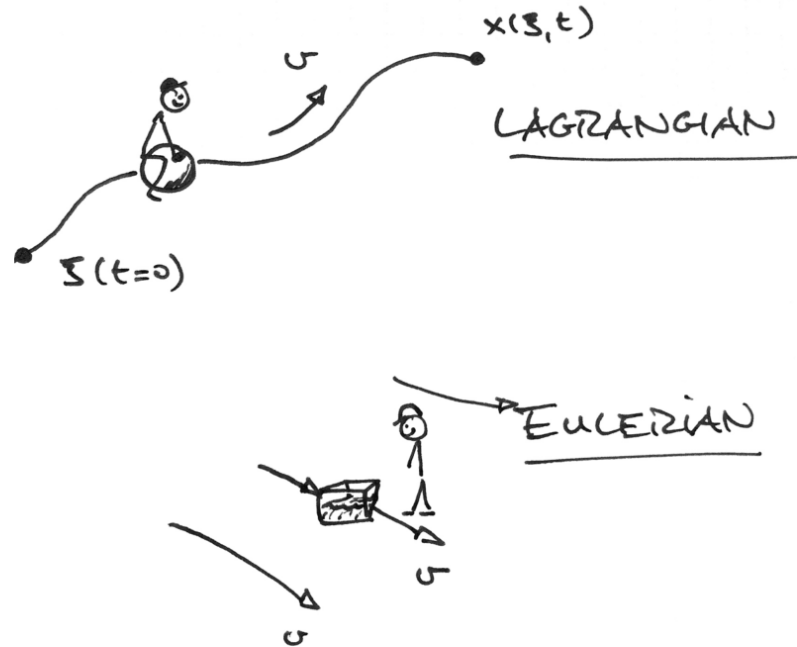


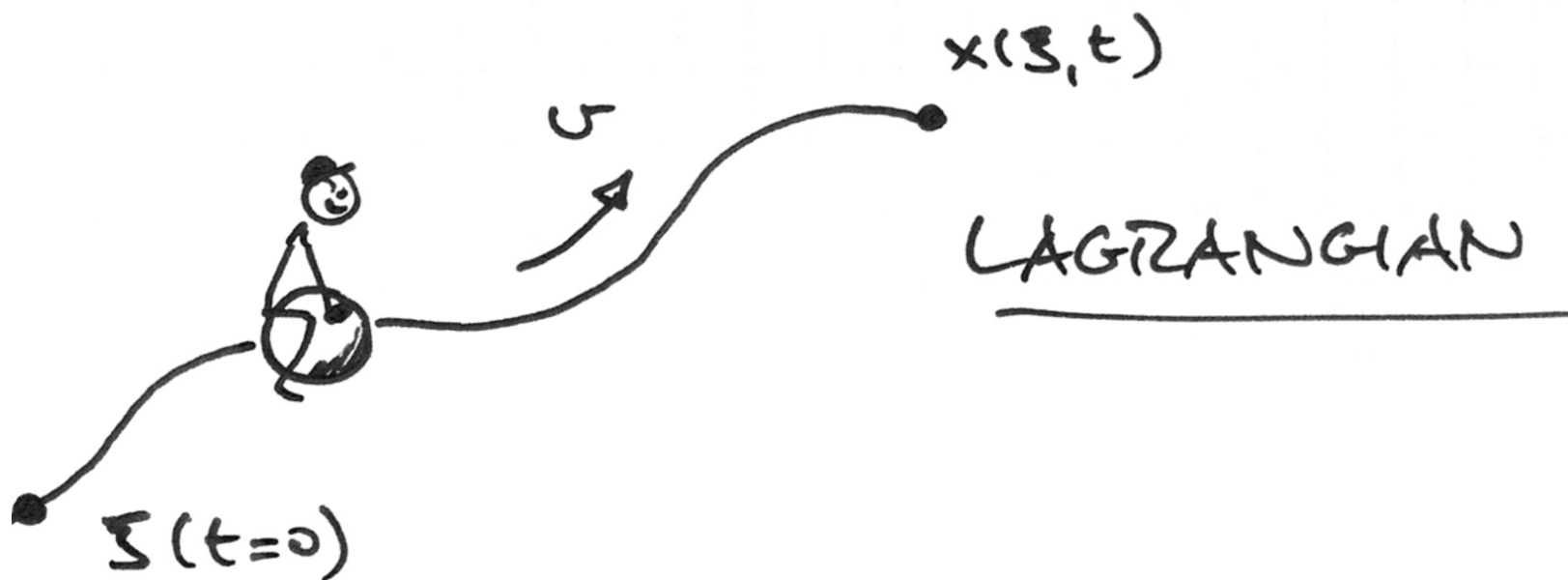
Differentialgleichungen II



Numerische Lösung der Transportgleichung:
Lagrange und Semi-Lagrange Verfahren

Lagrangesche Perspektive





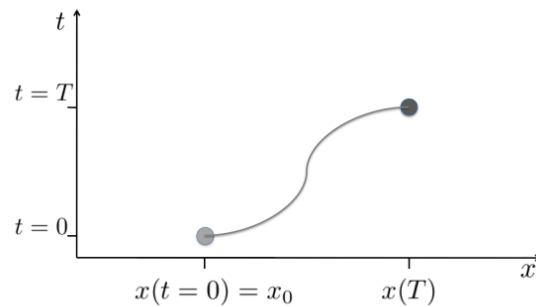
Formalisierung

- Position: $x = x(t)$,
- Geschwindigkeit: $v = v(x, t)$.

Partikel Position berechnet sich aus

$$\dot{x} = \frac{dx}{dt} = v(x, t)$$

Mit Anfangsbedingung $x(t = 0) = x_0$



Lagrangescher Transport mit Quellterm

Die Gleichung lautet

$$\frac{d\rho}{dt} = s(x, t)$$

mit $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot \frac{dx}{dt}$

Bemerkung: Mit $\dot{x} = v(x, t)$ erhalten wir

$$\frac{d\rho}{dt} = \rho_t + v\rho_x = s(x, t).$$

Zusammenfassung: Löse also zwei gewöhnliche DGLn:

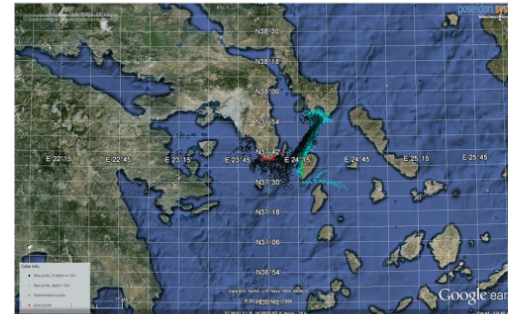
$$\begin{aligned}\dot{x} &= v(x, t), & x(0) &= x_0, \\ \dot{\rho} &= s(\rho, x, t), & \rho(x, 0) &= \rho_0(x).\end{aligned}$$

Bemerkung: Für Anwendungen werden viele Partikel verwendet!

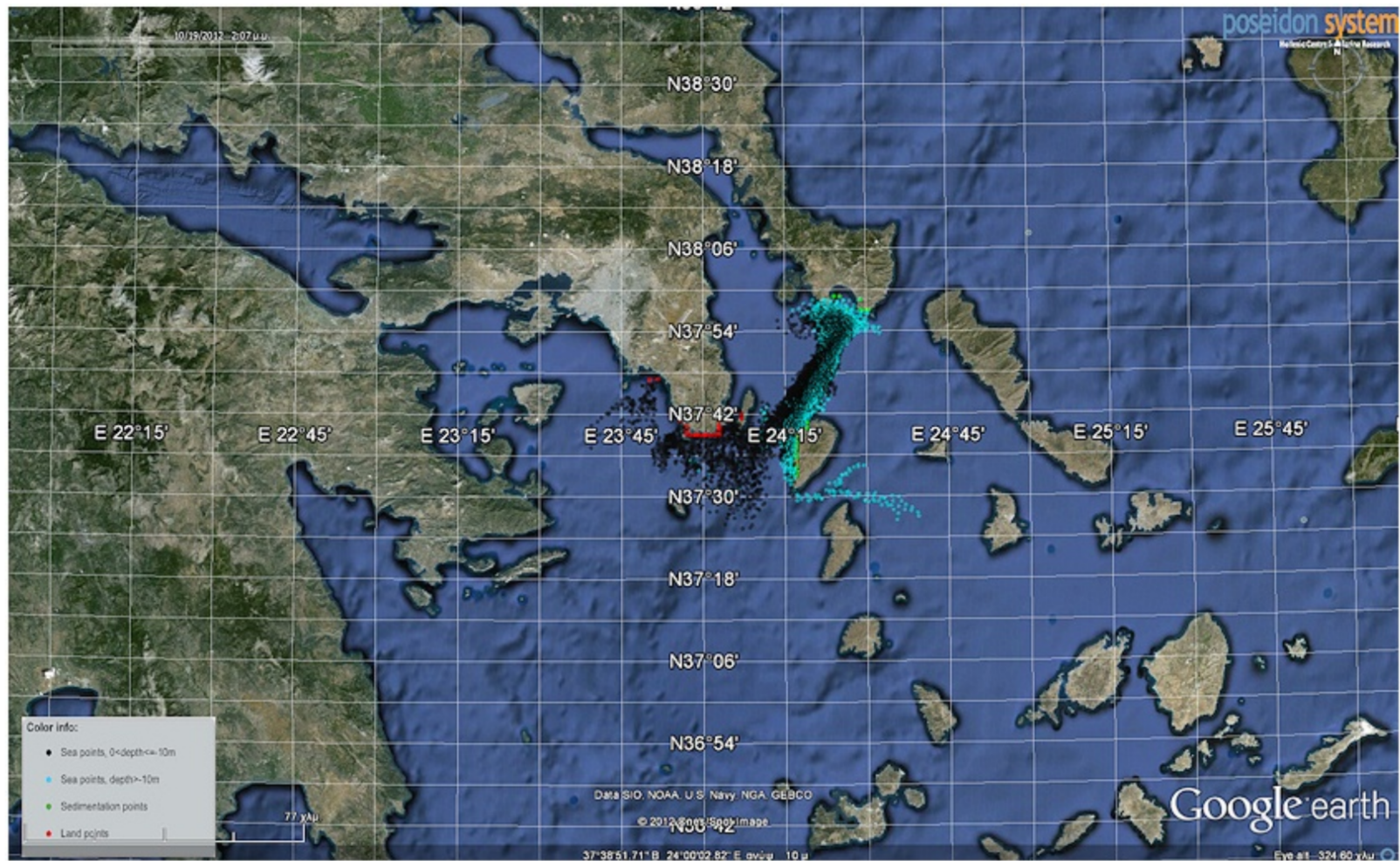
Bemerkung: Bei passiver Advektion ($s \equiv 0$) erhalte:

$$\dot{\rho} = 0 \quad \Rightarrow \quad \rho(x, t) \equiv \text{konst.} = \rho_0(x).$$

Da $x = x(t)$, erhalte parametrisierte Funktion: $\rho(x, t) = \rho(x(t), t)$.



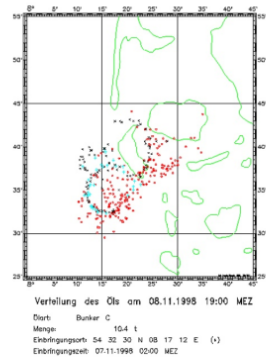
<http://www.medess4ms.eu/oil-spill-models>

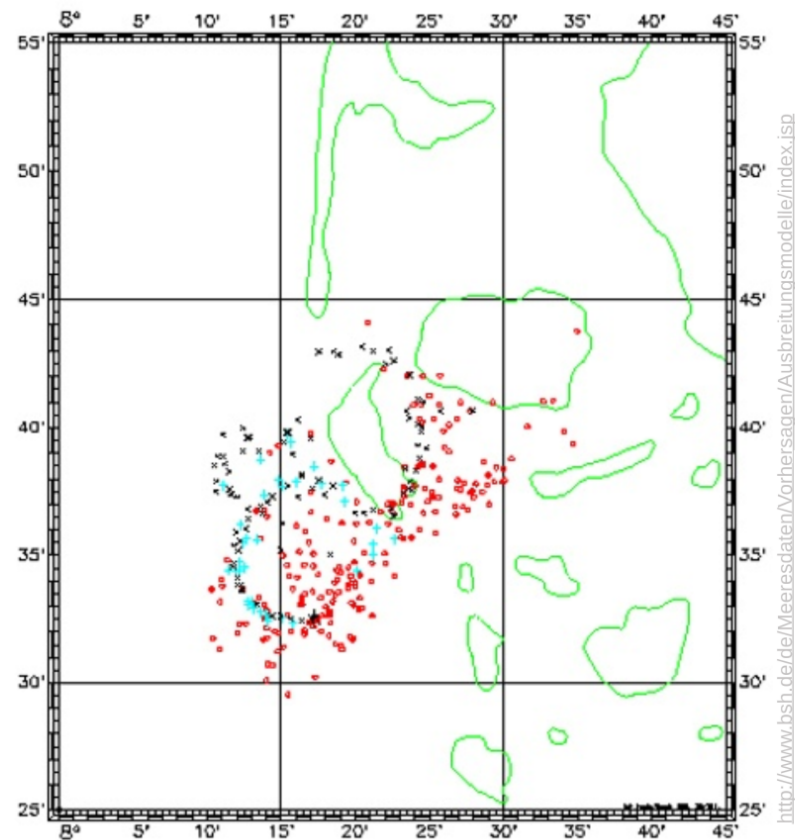


<http://www.medess4ms.eu/oil-spill-models>

Probleme rein Lagrangescher Methoden

- Verteilung der Partikel wird irregulär.
- Interaktion zwischen Partikeln schwierig zu simulieren (Diffusion).
- Dichteverteilung als räumliche Funktion schwierig zu Rekonstruieren.





Verteilung des Öls am 08.11.1998 19:00 MEZ

Ölart: Bunker C

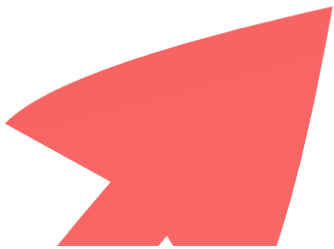
Menge: 10.4 t

Einbringungsort: 54 32 30 N 08 17 12 E (•)

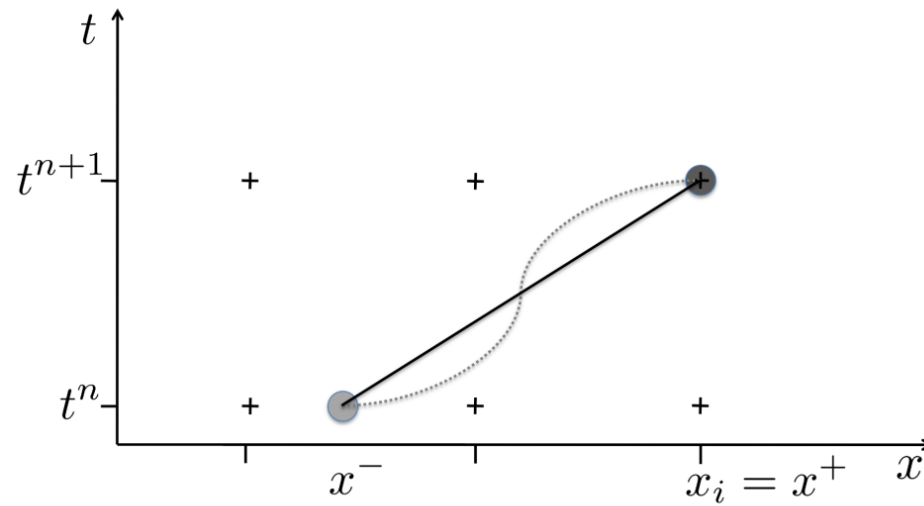
Einbringungszeit: 07.11.1998 02:00 MEZ

Idee

Kombiniere Lagrangesche und Eulersche Methode

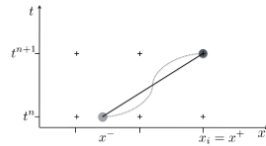


Idee der Semi-Lagrange Methode



Formalisierung

Problem: Passive Advektion ($s \equiv 0$):



$$\begin{aligned}\frac{dx}{dt} &= v(x, t), & x(0) &= x_0, \\ \frac{d\rho}{dt} &= 0, & \rho(x, 0) &= \rho_0(x).\end{aligned}$$

Strategie:

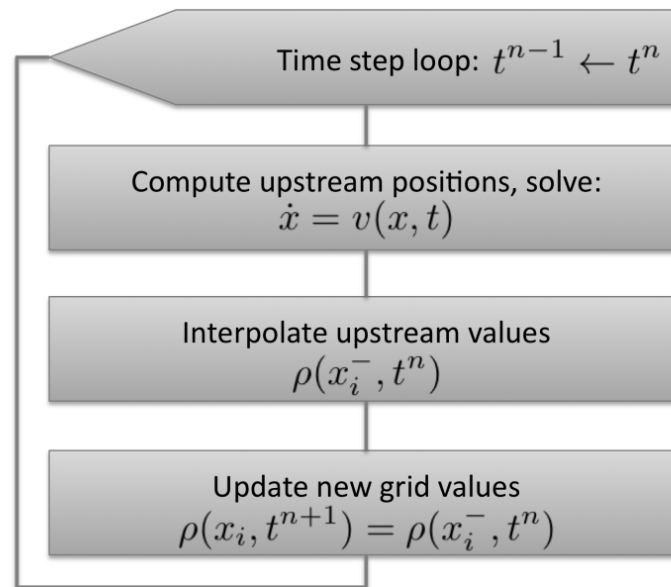
- Löse $\frac{dx}{dt} = v$ mit geeignetem Löser für gew. DGL,
- Löse $\frac{d\rho}{dt} = 0$ durch finite Differenzen.

$$\frac{d\rho}{dt} \approx \frac{\rho(x_i, t^{n+1}) - \rho(x_i^-, t^n)}{\Delta t} = 0$$

$$\Rightarrow \rho^+ = \rho^-.$$

$x_i, i = 1 : N$ grid points, $t^n, n = 1 : M$ time steps.

Algorithmus



Stabilität und Konsistenz

Von Neumann Stability Analysis:

Assume linear interpolation of upstream points and exact wind, i.e.

$$\rho_i^{n+1} = (1 - \nu)\rho_{k_i}^n + \nu\rho_{k_i-1}^n$$

- $[x_{k_i-1}, x_{k_i}]$ interval containing upstream point x_i^- ,
- $\nu = \frac{x_{k_i} - x_i^-}{\Delta x}$.

Then using $z_n e^{ik(jh)}$ we have:

$$\begin{aligned} z_{n+1} e^{ik(jh)} &= (1 - \nu)z_n e^{ik(k_i h)} + \nu z_n e^{ik(k_i - 1)h}, \\ &= z_n \left[1 - \nu(1 - e^{-ik(h)}) \right] e^{ik(k_i - j)h} e^{ik(jh)}; \\ \Rightarrow \xi &= \left[1 - \nu(1 - e^{-ik(h)}) \right] e^{ik(k_i - j)h}, \\ \Rightarrow |\xi|^2 &= 1 - 2\nu(1 - \nu)(1 - \cos(kh)). \end{aligned}$$

Stability follows for: $0 \leq \nu \leq 1$, i.e. always!

Remark (Order of Consistency):

It can be shown the the semi-Lagrangian advection scheme retains the consistency order of the discretization schemes involved:

$$\left. \begin{array}{l} \dot{x} = v \text{ order } p \\ \dot{\rho} = 0 \text{ order } p \end{array} \right\} \Rightarrow \text{SLM order } p.$$

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Then using $z_n e^{ik(jh)}$ we have:

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at wind, i.e.

- $[x_{k_i-1}, x_{k_i}]$ interval containing upstream point x_i^- ,
- $\nu = \frac{x_k - x^-}{\Delta x}$.

Then using $z_n e^{ik(jh)}$ we have:

$$\begin{aligned} z_{n+1} e^{ik(jh)} &= (1 - \nu) z_n e^{ik(k_i h)} + \nu z_n e^{ik(k_i - 1)h}, \\ &= z_n \left[1 - \nu(1 - e^{-ik(h)}) \right] e^{ik(k_i - j)h} e^{ik(jh)}, \\ \Rightarrow \xi &= \left[1 - \nu(1 - e^{-ik(h)}) \right] e^{ik(k_i - j)h}, \\ \Rightarrow |\xi|^2 &= 1 - 2\nu(1 - \nu)(1 - \cos(kh)). \end{aligned}$$

$$\begin{aligned}
z_{n+1} &= (1 - \nu)z_n \\
&= z_n \left[1 - \nu(1 - e) \right] \\
\Rightarrow \xi &= \left[1 - \nu(1 - e) \right] \\
\Rightarrow |\xi|^2 &= 1 - 2\nu(1 - \nu)(1 - e)
\end{aligned}$$

Stability follows for: $0 \leq \nu \leq 1$, i.e. always!

Remark (Order of Consistency):

It can be shown that the semi-Lagrangian scheme has the consistency order of the discretization.

for: $0 \leq \nu \leq 1$, i.e. always!

Remark (Order of Consistency):

It can be shown the the semi-Lagrangian advection scheme retains the consistency order of the discretization schemes involved:

$$\left. \begin{array}{l} \dot{x} = v \text{ order } p \\ \dot{\rho} = 0 \text{ order } p \end{array} \right\} \Rightarrow \text{SLM order } p.$$

Advektion mit Quellterm

Erinnerung:

$$\rho_t + v\rho_x = s(x, t).$$

Lagrange:

$$\dot{x} = v; \quad \dot{\rho} = s.$$

Für einen Partikel löse:

$$\begin{aligned}\dot{x} &= v(x, t), & x(0) &= x_0, \\ \dot{\rho} &= s(\rho, x, t), & \rho(x, 0) &= \rho_0(x).\end{aligned}$$

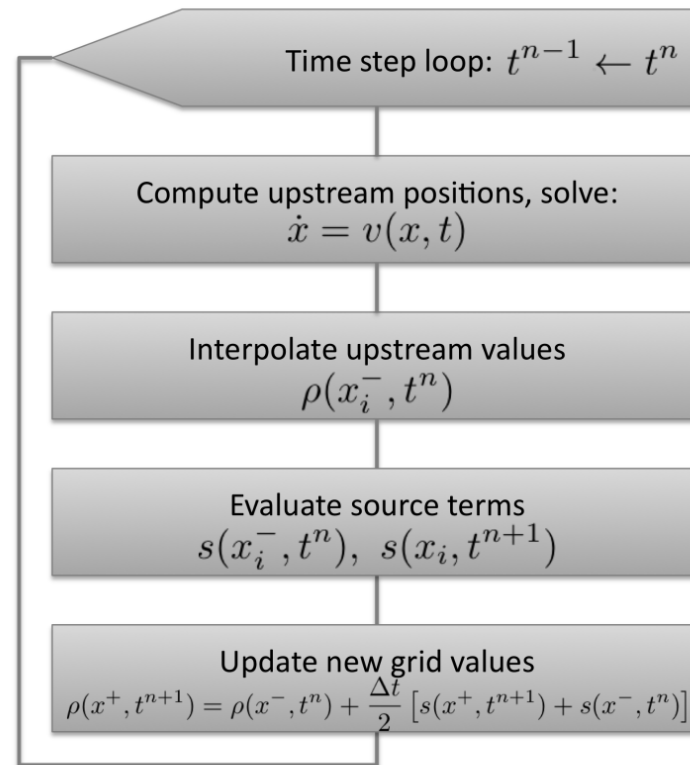
Annahme: $s = s(x, t)$ und verwende Trapezregel:

$$\frac{\rho(x^+, t^{n+1}) - \rho(x^-, t^n)}{\Delta t} = \frac{1}{2} [s(x^+, t^{n+1}) + s(x^-, t^n)]$$

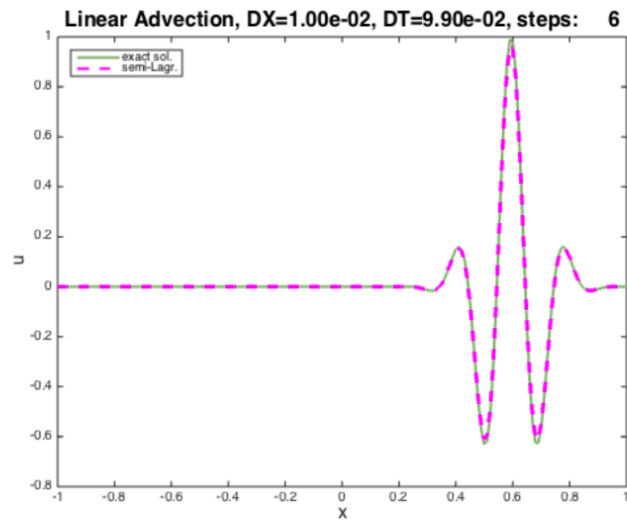
Oder einfach (Mittelpunktsregel):

$$\frac{\rho(x^+, t^{n+1}) - \rho(x^-, t^n)}{\Delta t} = s(x^0, t^{n+1/2})$$

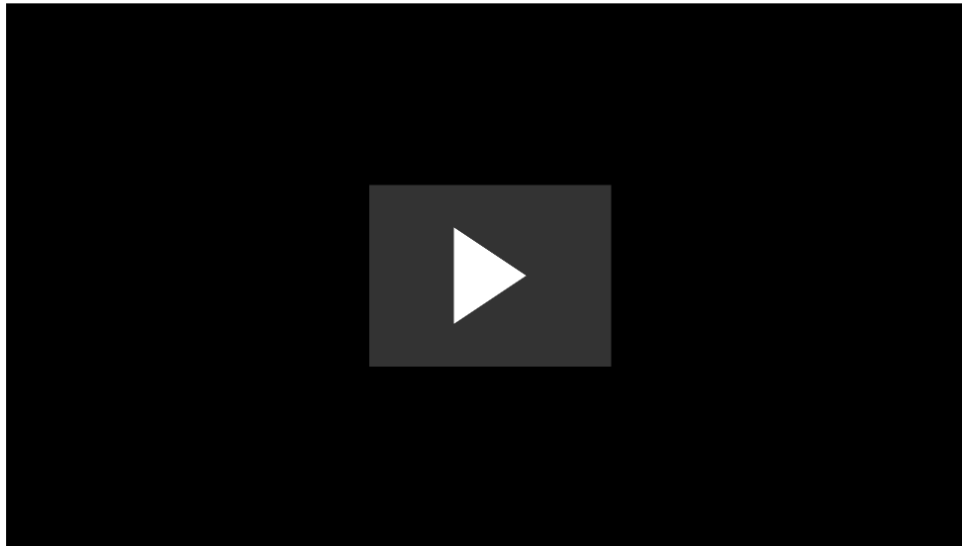
Algorithmus mit Quellterm



Numerisches Ergebnis



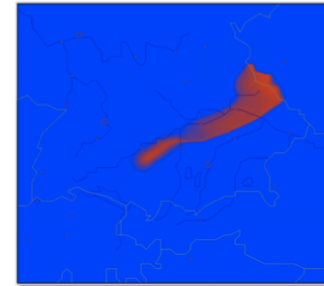
Jetzt 2D



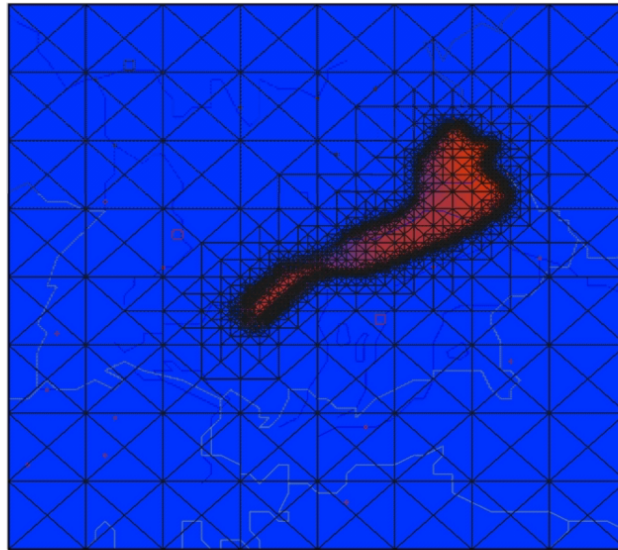
Rauchausbreitung als passiver Spurenstoff

Multi-Skalen Problem:

Gesamte Ausbreitung	$\mathcal{O}(10^5 m^2)$
Lokale Konzentrationen	$\mathcal{O}(10^2 m^2)$



Idee: Adaptive Gitterverfeinerung



- Verfeinerung nur dort wo notwendig
- Dynamische Adaption während Laufzeit

Advections-Diffusions Gleichung

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) + \nabla \cdot (\mu \nabla \rho) = 0$$

Spurenstoff (möglich: mehrere Komponenten)

$$\rho : \Omega \times T \rightarrow \mathbb{R}^m, \quad \Omega \subset \mathbb{R}^d$$

Gegebenes Wind Feld

$$\mathbf{v} : \Omega \times T \rightarrow \mathbb{R}^d$$

Gegebener Diffusionskoeffizient

$$\mu : \Omega \times T \rightarrow \mathbb{R}$$

Lagrangesche Form

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v} \rho) &= \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \nabla \cdot \mathbf{v} \rho \\ &= \frac{d\rho}{dt} + \nabla \cdot \mathbf{v} \rho.\end{aligned}$$

Divergenzfreier Fluss: $\nabla \cdot \mathbf{v} = 0$

$$\frac{d\rho}{dt} = 0$$

Splitting: Advektion und Diffusion

$$\frac{d\rho}{dt} + \nabla \cdot (\mu \nabla \rho) = 0$$

Advektion

Diffusion

Zeitintegration über einen Zeitschritt:

$$\int_t^{t+\Delta t} \frac{d\rho}{dt} + \nabla \cdot (\mu \nabla \rho) dt = 0$$

Advektionsterm (exakte Integration):

$$\int_t^{t+\Delta t} \frac{d\rho}{dt} dt = \rho(x, t + \Delta t) - \rho(x - \alpha, t)$$

Diffusionsterm (Trapezregel):

$$\int_t^{t+\Delta t} \nabla \cdot (\mu \nabla \rho) dt \approx \frac{1}{2} [\nabla \cdot (\mu^+ \nabla \rho^+) + \nabla \cdot (\mu^- \nabla \rho^-)] \cdot \Delta t$$

Diskrete Advektions-Diffusions Gleichung

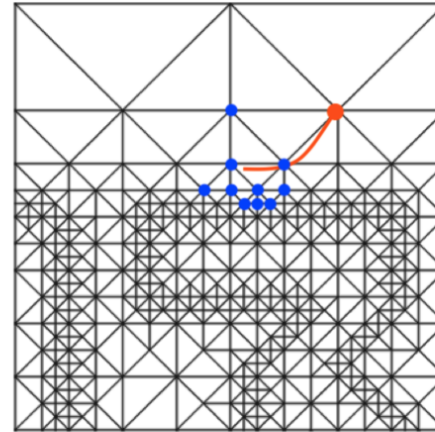
$$\rho^+ = \rho^- - \frac{\Delta t}{2} [\nabla \cdot (\mu^+ \nabla \rho^+) + \nabla \cdot (\mu^- \nabla \rho^-)] = 0$$

Problem bei nicht-uniformen Gittern

Semi-Lagrange-Methode:

Lagrange Form: $\frac{dc}{dt} = 0$

Differenz entlang Trajektorie:



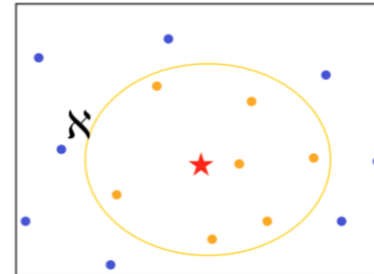
$$\frac{dc}{dt} \approx \frac{c(\vec{x}, t + \Delta t) - c(\vec{x} - 2\vec{\alpha}, t - \Delta t)}{2\Delta t}$$

$$\Rightarrow \boxed{c(\vec{x}, t + \Delta t)} = \boxed{c(\vec{x} - 2\vec{\alpha}, t - \Delta t)}$$

Interpolation mit radialen Basisfunktionen

Interpolationsproblem:

$$s : \mathbb{R}^d \rightarrow \mathbb{R} \quad \text{mit} \quad s|_{\mathbb{N}} = c(\cdot, t)|_{\mathbb{N}}.$$



$\mathbb{N}$ Menge der k Nachbarn

Interpolierende Funktion:

$$s(x) = \sum_{j=1}^k \lambda_j \Phi(\|x - y_j\|) + p(x), \quad y_j \in \mathbb{N}, \quad p(x) = \sum_{l=1}^q \mu_l p_l(x).$$

$$\begin{bmatrix} A_{\Phi, \mathbb{N}} & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} c(y_j) \\ 0 \end{bmatrix}$$

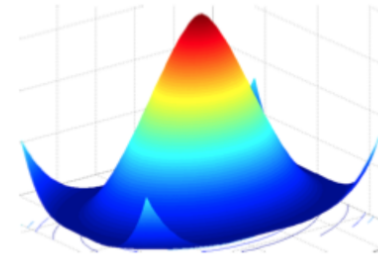
$$A_{\Phi, \mathbb{N}} = [\Phi(\|x_j - x_l\|)]_{j,l=1:k}, \quad P = [p_n(x_j)]_{n=1:q, j=1:k}.$$

($\mathbb{N}$ nicht-degeneriert)

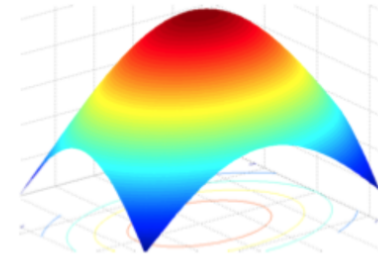
R.L. Hardy (1971), J. Duchon (1976), W.R. Madych/S.A. Nelson (1988)

Beispiele radialer Basisfunktionen

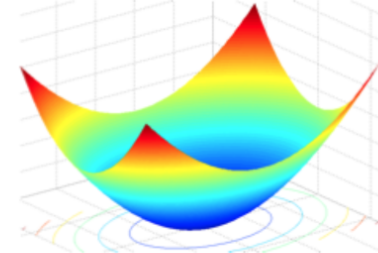
Thin Plate Spline $\Phi(r) = r^2 \log r$



Gaussians $\Phi(r) = e^{-r^2}$

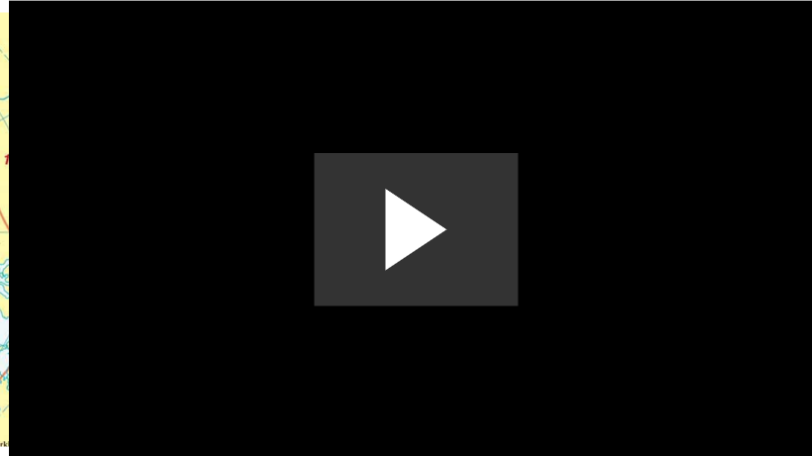


Multiquadrics $\Phi(r) = \sqrt{r^2 + 1}$



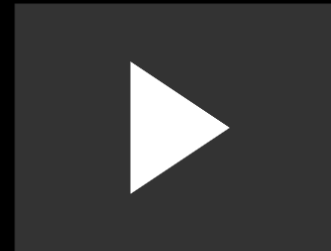
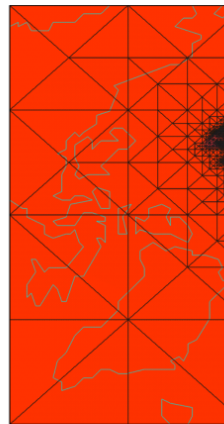
Anwendungsbeispiel: Spurenstofftransport

Gegeben: Wind-Daten



<http://www.lib.utexas.edu/maps>

Spurenstofftransport (adaptives Gitter)



Spurenstofftransport (gitterfrei)

