Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 7

Exercise 1:

Given the following initial boundary value problem for $u = u(x,t)$:

$$\begin{aligned}
u_{tt} - 4u_{xx} &= e^{-t}\left(1 - \frac{x}{3}\right) && x \in (0,\,3),\ t > 0,\\
u(x,0) &= 1 + 2\sin(\pi x) && x \in [0,3],\\
u_t(x,0) &= \frac{x}{3} && x \in (0,3), \qquad (1)\\
u(0,t) &= e^{-t} && t \geq 0,\\
u(3,t) &= 1 && t \geq 0.
\end{aligned}$$

a) Show that the homogenization of the boundary data according to

$$v = u - e^{-t} - \frac{x}{3}(1 - e^{-t})$$

leads to the following initial boundary value problem for v:

$$\begin{aligned}
v_{tt} - 4v_{xx} &= 0 && x \in (0,\,3),\ t > 0,\\
v(x,0) &= 2\sin(\pi x) && x \in [0,3],\\
v_t(x,0) &= 1 && x \in (0,3), \qquad (2)\\
v(0,t) &= 0 && t \geq 0,\\
v(3,t) &= 0 && t \geq 0.
\end{aligned}$$

b) Solve the initial boundary value problem (2) from part a) and compute the solution to the initial boundary value problem (1).

Solution:

a) With $v = u - e^{-t} - \dfrac{x}{3}(1 - e^{-t})$ it holds

$u_t = v_t - e^{-t}(1 - \dfrac{x}{3}),\quad u_{tt} = v_{tt} + e^{-t}(1 - \dfrac{x}{3}),\quad v_{xx} = u_{xx}.$

New differential equation:

$$v_{tt} + e^{-t}(1 - \frac{x}{3}) - 4v_{xx} = e^{-t}(1 - \frac{x}{3}) \iff v_{tt} - 4v_{xx} = 0\,.$$

New initial and boundary values:

$v(x,0) = u(x,0) - e^0 - \frac{x}{3}(1-e^0) = 1 + 2\sin(\pi x) - 1 = 2\sin(\pi x)$,

$v_t(x,0) = u_t(x,0) + e^0 - \frac{x}{3}\cdot e^0 = 1$,

$v(0,t) = v(3,t) = 0$.

b) Every function

$$v(x,t) = \sum_{k=1}^{n} (A_k\cos(ck\omega t) + B_k\sin(ck\omega t))\cdot\sin(k\omega x) \qquad \omega = \frac{\pi}{l}$$

with $c = 2$, $\omega = \frac{\pi}{3}$.

fulfills the boundary conditions and the differential equation.

The initial conditions remain to be fulfilled. The first one for $n \to \infty$:

$$v(x,0) = \sum_{k=1}^{\infty} (A_k\cos(0) + B_k\sin(0))\cdot\sin\left(\frac{k\pi x}{3}\right) = \sum_{k=1}^{\infty} A_k\sin\left(\frac{k\pi x}{3}\right) = 2\sin(\pi x) \qquad x \in [0,3]$$

The coefficients A_k can be obtained straightforwardly:

$A_3 = 2, \qquad A_k = 0 \quad \forall\, k \neq 3.$

The second initial condition requires:

$$v_t(x,0) = \sum_{k=1}^{\infty} ck\omega B_k\cdot\sin(k\pi x) = 1$$

With the Fourier coefficients of the (discontinuous) odd continuation of $v_t(x,0)$

$$b_k = \tfrac{2}{3}\int_0^3 \sin\left(\frac{k\pi x}{3}\right)\,dx = \frac{2}{3}\left.\frac{-\cos\left(\frac{k\pi x}{3}\right)}{\frac{k\pi}{3}}\right|_0^3 = \frac{2}{k\pi}(\cos(0) - \cos(k\pi))$$

one obtains

$B_k = \frac{3}{2k\pi}b_k = \frac{3}{2k\pi}\cdot\frac{2}{k\pi}(\cos(0)-\cos(k\pi)) = \frac{3}{k^2\pi^2}(1-\cos(k\pi))$.

So it holds:

$$v(x,t) = 2\cos(2\pi t)\sin(\pi x) + \sum_{k=1}^{\infty}\frac{3}{k^2\pi^2}(1-(-1)^k)\sin\left(\frac{2k\pi}{3}t\right)\cdot\sin\left(\frac{k\pi}{3}x\right).$$

The solution to the original problem is then

$u(x,t) = v(x,t) + e^{-t} + \frac{x}{3}(1-e^{-t})$.

Exercise 2:

For the numerical solution of a differential equation for $u(x,t)$, $x \in]0,(n+1)\Delta x[, t > 0$ with given initial data at $t = 0$ and boundary data at $x = 0$ and $x = (n+1)\Delta x$ the following grid is defined

$$x_j = j \cdot \Delta x, \qquad j = 0, 1, \ldots, n+1, \qquad\qquad t_m = m \cdot \Delta t, \qquad m = 0, 1, 2, \ldots$$

u_j^m is an approximation of $u(x_j, t_m)$.

Which PDEs are approximated by the following difference equations with the corresponding initial data ($m = 0$) and boundary data ($j = 0$ or $j = n+1$)?

For $j = 0, \ldots, N$ and $m = 1, 2, 3 \ldots$:

a) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^m - u_{j-1}^m}{\Delta x} = 0,$$

b) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^{m+1} - u_{j-1}^{m+1}}{\Delta x} = 0,$$

c) $$\frac{u_j^{m+1} - 2u_j^m + u_j^{m-1}}{\Delta t^2} = \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2},$$

d) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} = \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2}$$

e) $$\frac{u_j^{m+1} - u_j^m}{\Delta t} + c\,\frac{u_j^m - u_{j-1}^m}{\Delta x} = \frac{u_{j+1}^{m+1} - 2u_j^{m+1} + u_{j-1}^{m+1}}{\Delta x^2}$$

For which difference equations can the data at time point $m + 1$ be calculated directly if the data at time m is known? So which method is an explicit method?

Solution:

a) $u_t + cu_x = 0$, explicit.

b) $u_t + cu_x = 0$, implicit.

c) $u_{tt} = u_{xx}$, implicit.

d) $u_t + cu_x = u_{xx}$, explicit.

e) $u_t + cu_x = u_{xx}$, implicit.

Discussion: 11.07. – 15.07.2022