Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 7

Exercise 1: (Vibrating String)

Solve the initial boundary value problem

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} && \text{for} \quad 0 < x < 1,\ t > 0,\\
u(0,t) &= u(1,t) = 0 && \text{for} \quad t > 0,\\
u(x,0) &= 0 && \text{for} \quad 0 < x < 1,\\
u_t(x,0) &= \begin{cases} 1, & \frac{1}{20} \le x \le \frac{1}{10},\\ 0 & \text{else,}\end{cases}
\end{aligned}$$

using the suitable product ansatz.
You will get a Fourier series as the solution. Plot the partial sums of the first 20 non-vanishing summands of this series for $c = 2$, $x \in [0,1], t \in [0, 0.4]$ and $t \in [0, 2]$.

Solution:

Product ansatz $u(x,t) = X(x) \cdot T(t)$ gives us $X(x) \cdot \ddot{T}(t) = c^2 X''(x) \cdot T(t)$,

Hence we have : $c^2 \dfrac{X''}{X} = \dfrac{\ddot{T}}{T} = -\lambda c^2$, $X'' = -\lambda X$ and $\ddot{T} = -\lambda c^2 T$

The homogeneous boundary conditions yield

$u(0,t) = X(0)T(t) = 0 \quad \forall t > 0 \quad \Longrightarrow \quad X(0) = 0 \vee T \equiv 0$

$u(1,t) = X(1)T(t) = 0 \quad \forall t > 0 \quad \Longrightarrow \quad X(1) = 0 \vee T \equiv 0$

For the nontrivial solution we obtain the following boundary value problem:

$$X''(x) = -\lambda X(x), \qquad X(0) = X(1) = 0$$

with the already familiar solutions:

$$\boxed{X_k(x) = \sin(k\omega x) \qquad \omega = \pi/1, \quad \lambda_k = \left(\frac{k\pi}{1}\right)^2 = (k\omega)^2, \quad k \in \mathbb{N}}$$

For T we have $\ddot{T} = -\lambda c^2 T = -(ck\omega)^2 T$

$T_k(t) = A_k \cos(ck\omega t) + B_k \sin(ck\omega t)$

Every function $u_k(x,t) := T_k(t) \cdot X_k(x)$ solves the homogeneous wave equation and fulfills the boundary values. Hence all linear combinations of these functions do as well:

$$\boxed{u(x,t) = \sum_{k=1}^{n} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)}$$

The initial conditions remain to be fulfilled. First one for $n \to \infty$ yields

$$u(x,0) = \sum_{k=1}^{\infty} \left(A_k \cos(0) + B_k \sin(0)\right) \cdot \sin(k\omega x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) = 0 \qquad x \in [0,1]$$

So we obtain $A_k = 0, \forall k \in \mathbb{N}$.

From the second initial condition we have:

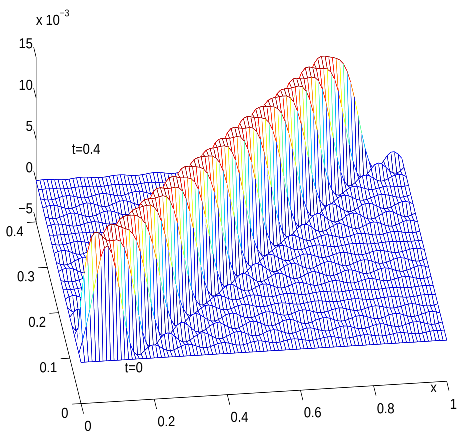
$$u_t(x,0) = \sum_{k=1}^{\infty} ck\pi B_k \cdot \sin(k\pi x) = v_0(x) = \begin{cases} 1, & \frac{1}{20} \leq x \leq \frac{1}{10} \\ 0 & \text{else} \end{cases} \qquad x \in [0,1]$$

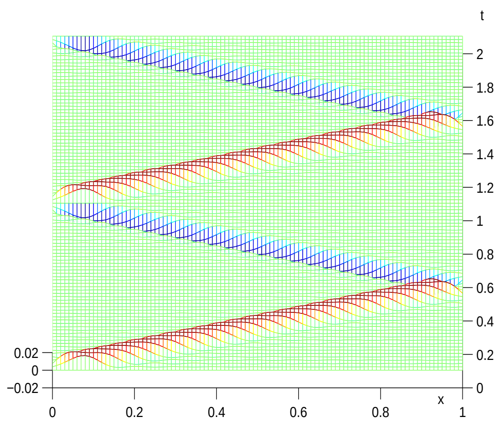
We compute the Fourier coefficients of the odd, 2-periodic continuation of v_0:

$$b_k = 2 \int_0^1 v_0(x) \sin(k\pi x) dx = 2 \int_{\frac{1}{20}}^{\frac{1}{10}} \sin(k\pi x) dx = \frac{2}{k\pi} \left[\cos\left(\frac{k\pi}{20}\right) - \cos\left(\frac{k\pi}{10}\right)\right]$$

With $B_k = \dfrac{b_k}{ck\pi}$ we have

$$\boxed{u(x,t) = \frac{2}{c\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \left[\cos\left(\frac{k\pi}{20}\right) - \cos\left(\frac{k\pi}{10}\right)\right] \sin(ck\omega t) \cdot \sin(k\omega x)}$$





Exercise 2:

We are looking for an approximation of the solution to the following problem

$$u_{tt} = u_{xx} \qquad x \in (0, 2\pi),\ t > 0,$$

$$u(x,0) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} < x < 2\pi \end{cases}$$

$$u_t(x,0) = 0 \qquad x \in (0, 2\pi)$$

$$u(0,t) = u(2\pi, t) = 0 \qquad t > 0$$

Sketch the 2π-periodic continuation of the initial data for $x \in [-2\pi, 4\pi]$.

Determine an approximation $\tilde{u}$ to the solution u of the problem using first three terms of the Fourier series.

Check which boundary and initial conditions are already fulfilled by this approximate solution.

Solution:

General solution:

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)}$$

With $\omega = \frac{\pi}{L} = \frac{\pi}{2\pi}$ and $c = 1$. Hence

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(\frac{k}{2}\, t) + B_k \sin(\frac{k}{2}\, t)\right) \cdot \sin(\frac{k}{2}\, x)$$

$$u_t(x,0) = \sum_{k=1}^{\infty} \frac{k}{2} \left(-A_k \sin(0) + B_k \cos(0)\right) \cdot \sin(\frac{k}{2}\, x) \overset{!}{=} 0 \implies B_k = 0\,,\ \forall k$$

$$u(x,t) = \sum_{k=1}^{\infty} A_K \cos(\frac{k}{2}\, t) \sin(\frac{k}{2}\, x)$$

with $u(x,0) = \sum_{k=1}^{\infty} A_K \sin(\frac{k}{2}\, x) \overset{!}{=} \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} < x < 2\pi \end{cases}$

So we have:

$$
\begin{aligned}
A_k &= \frac{1}{\pi}\int_0^{2\pi} u_0(x)\sin(\frac{k}{2}\,x)\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi/2} x\sin(\frac{k}{2}\,x)\,dx \;+\; \int_{\pi/2}^{3\pi/2}(\pi-x)\sin(\frac{k}{2}x)\,dx \;+\; \int_{3\pi/2}^{2\pi}(x-2\pi)\sin(\frac{k}{2}x)\,dx\right] \\
&= \frac{1}{\pi}\left(\left[-x\frac{\cos(\frac{k}{2}x)}{\frac{k}{2}}\right]_0^{\frac{\pi}{2}} + \int_0^{\pi/2}\frac{2}{k}\cos(\frac{k}{2}x)\;dx\right. \\
&+ \left[-(\pi-x)\frac{2}{k}\cos(\frac{k}{2}x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \left[(\frac{2}{k})^2\sin(\frac{k}{2}x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&- \left.\left[(x-2\pi)\frac{2}{k}\cos(\frac{k}{2}x)\right]_{\frac{3\pi}{2}}^{2\pi} + \left[(\frac{2}{k})^2\sin(\frac{k}{2}x)\right]_{\frac{3\pi}{2}}^{2\pi}\right) \\
&= \frac{8}{\pi k^2}\left(\sin(\frac{k\pi}{4}) - \sin(\frac{3k\pi}{4})\right).
\end{aligned}
$$

The first three coefficients are the following

$$
\begin{aligned}
A_1 &= \frac{8}{\pi}\left(\sin(\frac{\pi}{4}) - \sin(\frac{3\pi}{4})\right) \;=\; 0, \\
A_2 &= \frac{8}{4\pi}\left(\sin(\frac{\pi}{2}) - \sin(\frac{3\pi}{2})\right) \;=\; \frac{4}{\pi}, \\
A_3 &= \frac{8}{9\pi}\left(\sin(\frac{3\pi}{4}) - \sin(\frac{9\pi}{4})\right) \;=\; 0.
\end{aligned}
$$

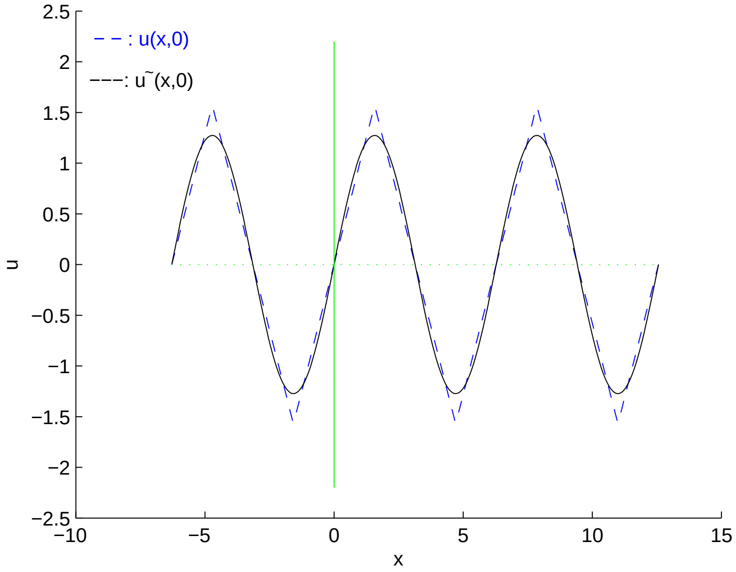
The approximation using the first three terms is given by

$$\tilde{u}(x,t) = u_3(x,t) = \frac{4}{\pi}\cos(t)\sin(x).$$

We have $u_3(x,0) = \frac{4}{\pi}\sin(x)$.

The first initial condition for u is hence fulfilled only approximately (see plot). The second initial condition is fulfilled as well as the boundary conditions.

Regarding the boundary conditions, any other outcome would have been a sure indication of a calculation error!



Discussion: 11.07.-15.07.2022