Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 6

Exercise 1:

From Lecture 9 we know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\,\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha$$

for solving the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = f(x),\ \hat{u}_t(x,0) = g(x),\ x \in \mathbb{R},\, c > 0.$$

The function

$$\tilde{u}(x,t) \;=\; \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau \tag{1}$$

solves the following initial value problem

$$\tilde{u}_{tt} \;-\; c^2\tilde{u}_{xx} \;=\; h(x,t) \qquad \tilde{u}(x,0) \;=\; \tilde{u}_t(x,0) \;=\; 0\,. \tag{2}$$

(Proof: Leibniz formula for the derivation of parameter-dependent integrals)

The initial value problem is to be solved

$$u_{tt} \;-\; 4u_{xx} \;=\; 6x\sin t, \qquad x \in \mathbb{R},\, t > 0$$
$$u(x,0) \;=\; x,\, x \in \mathbb{R}, \qquad u_t(x,0) \;=\; \sin(x),\, x \in \mathbb{R}$$

a) Compute the solution $\hat{u}$ to the IVP

$$\hat{u}_{tt} \;-\; 4\hat{u}_{xx} \;=\; 0, \qquad x \in \mathbb{R},\, t > 0$$
$$\hat{u}(x,0) \;=\; x,\, x \in \mathbb{R}, \qquad \hat{u}_t(x,0) \;=\; \sin(x),\, x \in \mathbb{R}.$$

b) Compute the solution $\tilde{u}$ to the IVP

$$\tilde{u}_{tt} \;-\; 4\tilde{u}_{xx} \;=\; 6x\sin t, \qquad x \in \mathbb{R},\, t > 0$$
$$\tilde{u}(x,0) \;=\; 0,\, x \in \mathbb{R}, \qquad \tilde{u}_t(x,0) \;=\; 0,\, x \in \mathbb{R}$$

c) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (2).

Solution:

a) Solution of the homogeneous differential equation with the non-homogeneous initial values following d'Alembert formula

$$\hat{u}(x,t) = \frac{1}{2}(x+2t+x-2t) + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(\eta)d\eta = x + \frac{1}{2}\sin(x)\sin(2t)$$

b) The solution of the original problem consists of the two partial solutions

c) solution of the non-homogeneous differential equation with homogeneous initial values

$$\begin{aligned}\tilde{u}(x,t) &= \frac{1}{4}\int_0^t \int_{x+2(\tau-t)}^{x-2(\tau-t)} 6\omega\,\sin(\tau)\,\mathrm{d}\omega\mathrm{d}\tau = \frac{3}{4}\int_0^t \sin(\tau)\,\left[\omega^2\right]_{x+2(\tau-t)}^{x-2(\tau-t)}\mathrm{d}\tau \\ &= \frac{3}{4}\int_0^t \sin(\tau)\,(-8x(\tau-t)\mathrm{d}\tau = 6x\int_0^t t\sin(\tau) - \tau\sin(\tau)\mathrm{d}\tau \\ &= 6xt(1-cos(t)) + 6x\left[\tau\cos(\tau)\right]_0^t - 6x\int_0^t \cos(\tau)\mathrm{d}\tau = 6x(t-\sin t)\,.\end{aligned}$$

together with:

$$u(x,t) = 6x(t-\sin t) + x + \frac{1}{2}\sin(x)\sin(2t)$$

Test:

$$u(x,0) = 6x(0-\sin(0)) + x + \frac{1}{2}\sin(x)\sin(0) = x,$$
$$u_t(x,t) = 6x(1-\cos(t)) + 0 + \frac{1}{2}\sin(x)2\cos(2t) = 6x(1-\cos(t)) + \sin(x)\cos(2t)$$
$$u_t(x,0) = 6x(1-\cos(0)) + \sin(x)\cos(0) = \sin(x)$$

$u_{xx} = 0 + 0 - \frac{1}{2}\sin(x)\sin(2t)$ (since the first two summands of u are linear in x, one only has to derive the sine term twice).

$$u_{tt} = 6x\sin t - 2)\,\sin(x)\sin(2t)$$
$$u_{tt} - 4u_{xx} = 6x\sin t - 2\,\sin(x)\sin(2t) - 4\cdot(-\tfrac{1}{2}\,\sin(x)\sin(2t)) = 6x\sin t\,.$$

Exercise 2:

a) Using a product ansatz, derive the series representation given in lecture 10 (page 18) for the solution of the following Neumann problem.

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1,\ t > 0, \\ u(x,0) &= g(x), & 0 < x < 1, \\ u_x(0,t) &= u_x(1,t) = 0 & t > 0. \end{aligned}$$

b) Solve the initial boundary value problem a) with $g(x) = 2\pi x - \sin(2\pi x)$.

Hint: $2\sin(\alpha) \cdot \cos(\beta) = \sin(\alpha + \beta) + \sin(\alpha - \beta)$.

Solution:

a) Short version: From the lecture we know that the ansatz $u_k(x,t) = v_k(x) \cdot w_k(t)$ with $L = 1$ leads to

$$v_k(x) = \cos(k\pi x), \qquad \text{and} \qquad w_k(t) = e^{-k^2\pi^2 t}, \qquad k \in \mathbb{N}_0$$

Very long version: The ansatz $u(x,t) = v(x) \cdot w(t)$ *yields:*

$v'' = -\lambda\, v, \quad \dot{w} = -\lambda w, \quad v'(0) = v'(1) = 0\,.$

Case distinction under the condition that the solution does not vanish:

$$\begin{aligned} \lambda = 0 &\Longrightarrow v(x) = a_0 + b_0 x, \quad v' = b_0 = 0 \\ &\Longrightarrow v_0(x) = a_0\,. \\ \lambda < 0 &\Longrightarrow v(x) = a e^{\sqrt{-\lambda}x} + b e^{-\sqrt{-\lambda}x} \\ &\quad v'(0) = 0 \Longleftrightarrow a = b \\ &\quad v'(1) = 0 \Longleftrightarrow a\sqrt{-\lambda}(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}) = 0 \\ &\quad \Longleftrightarrow (u \equiv 0) \vee (e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}} \Longleftrightarrow \lambda = 0) \qquad \textit{Contradiction!} \\ \lambda > 0 &\Longrightarrow v(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \\ &\quad v'(x) = (\sqrt{\lambda})(-a\sin(\sqrt{\lambda}\,x) + b\cos(\sqrt{\lambda}\,x) \\ &\quad v'(0) = 0 \Longleftrightarrow b = 0 \\ &\quad v'(1) = 0 \Longleftrightarrow (u \equiv 0) \vee (\sin(\sqrt{\lambda}) = 0 \Longleftrightarrow \lambda_k = k^2\pi^2). \end{aligned}$$

So overall we get

$$v_k(x) = \cos(k\pi x), \quad k \in \mathbb{N}_0\,.$$

One can easily calculate for the time component

$$w_k(t) = e^{-k^2\pi^2 t}, \quad k \in \mathbb{N}_0\,.$$

So as a series representation for the solution one has

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t} \cos(k\pi x)\,.$$

To fulfill:

$$u(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)\,.$$

To determine the coefficients, g is continued evenly and $2-$ periodically and the Fourier coefficients are determined

$$a_k = 2\int_0^1 g(x)\cos(k\pi x)\,dx\,.$$

b) For $k \notin \{0,2\}$ one computes for $g(x) = 2\pi x - \sin(2\pi x)$.

$$\begin{aligned}
a_k &= 2\int_0^1 (2\pi x - \sin(2\pi x))\cos(k\pi x)\,dx \\
&= 2\int_0^1 2\pi x\cos(k\pi x)\,dx - 2\int_0^1 \sin(2\pi x)\cos(k\pi x)\,dx \\
&= 4\pi x\,\frac{\sin(k\pi x)}{k\pi}\bigg|_0^1 - 4\pi\int_0^1 \frac{\sin(k\pi x)}{k\pi}\,dx - \int_0^1 \sin(2\pi x + k\pi x) + \sin(2\pi x - k\pi x)\,dx \\
&= \frac{4}{k^2\pi}\cos(k\pi x)|_0^1 + \frac{\cos((k+2)\pi x)}{(k+2)\pi}\bigg|_0^1 + \frac{\cos((-k+2)\pi x)}{(-k+2)\pi}\bigg|_0^1 \\
&= \frac{4}{k^2\pi}(\cos(k\pi)-1) + \frac{1}{(k+2)\pi}(\cos((k+2)\pi)-1) - \left(\frac{1}{(k-2)\pi}\cos((-k+2)\pi)-1\right) \\
&= \frac{4}{k^2\pi}\left((-1)^k - 1\right) - \left(\frac{1}{(k-2)\pi} - \frac{1}{(k+2)\pi}\right)(\cos(k\pi)-1) \\
&= \left(\frac{4}{k^2\pi} - \frac{4}{(k^2-4)\pi}\right)\cdot\left((-1)^k - 1\right) = \left(\frac{16\cdot\left(1-(-1)^k\right)}{k^2\cdot(k^2-4)\pi}\right)
\end{aligned}$$

For $k = 0$ we have

$$a_0 = 2\int_0^1 2\pi x - \sin(2\pi x)\,dx \;=\; 2\pi$$

and for $k = 2$

$$\begin{aligned}
a_2 &= 2\int_0^1 2\pi x\cos(2\pi x) - \sin(2\pi x)\cos(2\pi x)dx \\
&= \frac{4}{2^2\pi}\left((-1)^2 - 1\right) - \int_0^1 \sin(4\pi x)dx = 0.
\end{aligned}$$