Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 6

Exercise 1:

a) Solve the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && \text{on } \mathbb{R}^2\,, \\ u(x,0) &= 2\sin(4\pi x) && x \in \mathbb{R}\,, \\ u_t(x,0) &= \cos(\pi x) && x \in \mathbb{R}\,. \end{aligned}$$

b) Given the problem

$$u_{tt} = 9u_{xx}, \quad \text{for } x \in \mathbb{R},\ t > 0\,,$$

$$u(x,0) = f(x) = \begin{cases} 2 & -1 \le x \le 1\,, \\ 0 & \text{otherwise,} \end{cases}$$

$$u_t(x,0) = 0.$$

Sketch the obtained solution using d'Alembert's formula for

$t = 0,\ \frac{1}{6},\ \frac{1}{3},\ \frac{2}{3},\ 1\,.$

Solution:

a) Using d'Alembert we have

$$\begin{aligned} u(x,t) &= \sin\left(4\pi(x+t)\right) + \sin\left(4\pi(x-t)\right) + \frac{1}{2c}\int_{x-t}^{x+t} \cos(\pi\eta)d\eta \\ &= 2\sin(4\pi x)\cos(4\pi t) + \left.\frac{\sin(\pi\eta)}{2\pi}\right|_{x-t}^{x+t} \\ &= 2\sin(4\pi x)\cos(4\pi t) + \frac{1}{2\pi}\left(\sin\left(\pi(x+t)\right) - \sin\left(\pi(x-t)\right)\right) \\ &= 2\sin(4\pi x)\cos(4\pi t) + \frac{1}{\pi}\left(\cos\left(\pi x\right)\cdot\sin\left(\pi t\right)\right). \end{aligned}$$

b) D'Alembert's formula yields $u(x,t) = \dfrac{1}{2}\left(f(x+3t) + f(x-3t)\right)$. So we obtain

$$u(x,t) = \begin{cases} 2 & \text{if } -1 \le x - 3t \le 1 \text{ and } -1 \le x + 3t \le 1 \\ 1 & \text{if } -1 \le x - 3t \le 1 \text{ or } -1 \le x + 3t \le 1 \text{ with exclusive or} \\ 0 & \text{otherwise} \end{cases}$$

For example, one obtains for $t = 1/6$:

$x - 3t = x - 0.5 \in [-1, 1] \iff x \in [-0.5, 1.5]$ and

$x + 3t = x + 0.5 \in [-1, 1] \iff x \in [-1.5, 0.5]$

and hence

$$u(x, \frac{1}{6}) = \begin{cases} 2 & \text{for } x \in [-0.5, 0.5] \\ 1 & \text{for } x \in [-1.5, -0.5) \text{ or } x \in (0.5, 1.5] \\ 0 & \text{otherwise} \end{cases}$$

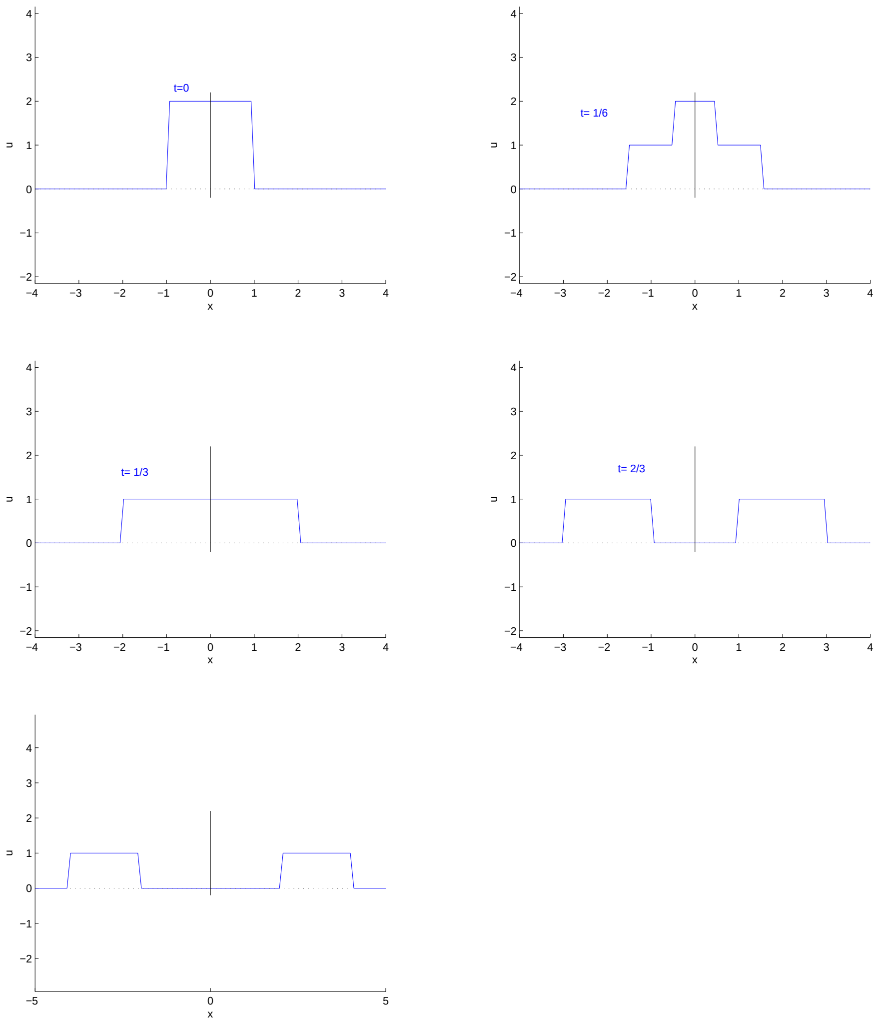
The solution for the other t values is calculated analogously.

$$u(x, \frac{1}{3}) = \begin{cases} 2 & \text{for } x = 0 \\ 1 & \text{for } x \in [-2, 0) \text{ or } x \in (0, 2] \\ 0 & \text{otherwise} \end{cases}$$

$$u(x, \frac{2}{3}) = \begin{cases} 1 & \text{for } x \in [-3, -1] \text{ or } x \in [1, 3] \\ 0 & \text{otherwise} \end{cases}$$

$$u(x, 1) = \begin{cases} 1 & \text{for } x \in [-4, -2] \text{ or } x \in [2, 4] \\ 0 & \text{otherwise} \end{cases}$$

The original (angular) wave clearly breaks up into two waves running in opposite directions.



Exercise 2:

The following problem is given for $u(x, y, t)$.

$$\begin{aligned} u_t &= u_{xx} + u_{yy}, & x, y \in (0, \pi),\ t > 0, \\ u(0, y, t)) &= u(\pi, y, t) = 0, & \text{for } y \in (0, \pi),\ t > 0, \\ u(x, 0, t)) &= u(x, \pi, t) = 0, & \text{for } x \in (0, \pi),\ t > 0, \\ u(x, y, 0) &= \frac{1}{2}(\sin(2x) + \sin(x))\sin(y) & \text{for } x, y \in (0, \pi). \end{aligned}$$

a) Using the ansatz $u(x, y, t) = T(t) \cdot X(x) \cdot Y(y)$ for the solution of the differential equation, derive three decoupled ordinary differential equations for X, Y and T.

b) Derive first from the boundary values

$$\begin{aligned} u(0, y, t) = u(\pi, y, t) = 0, &\quad \text{for } y \in [0, \pi],\ t > 0, \\ u(x, 0, t) = u(x, \pi, t) = 0, &\quad \text{for } x \in [0, \pi],\ t > 0, \end{aligned}$$

the boundary conditions for the solutions of the differential equations for X and Y, and solve the obtained ordinary boundary value problems for X and Y.

Then determine the appropriate functions $T(t)$.

c) Determine a series representation of the solution u to the original problem and fit it to the initial values

$$u(x, y, 0) = \frac{1}{2}(\sin(2x) + \sin(x))\sin(y) \quad \text{for } x, y \in [0, \pi].$$

How does the solution behave for $t \to \infty$?

Solution:

a) We use the ansatz $u(x, y, t) = T(t)X(x)Y(y)$. Hence we get:

$$\dot{T} \cdot X \cdot Y = T(X'' \cdot Y + X \cdot Y'')$$

$$\frac{\dot{T}}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\frac{\dot{T}}{T} = -\lambda \qquad \frac{X''}{X} = -\lambda - \frac{Y''}{Y} = -\mu$$

b) From the boundary conditions one obtains for non-identical vanishing solutions

$$\begin{aligned} u(0, y, t) &= X(0) \cdot Y(y) \cdot T(t) = 0 \Longrightarrow X(0) = 0\,, \\ u(\pi, y, t) &= X(\pi) \cdot Y(y) \cdot T(t) = 0 \Longrightarrow X(\pi) = 0\,, \\ u(x, 0, t) &= X(x) \cdot Y(0) \cdot T(t) = 0 \Longrightarrow Y(0) = 0\,, \\ u(x, \pi, t) &= X(x) \cdot Y(\pi) \cdot T(t) = 0 \Longrightarrow Y(\pi) = 0\,. \end{aligned}$$

For X one has

$$\frac{X''}{X} = -\mu\,, X(0) = X(\pi) = 0\,.$$

The solutions of this BVP have already been derived several times. They are

$$X_k(x) = a_k \sin(kx), \qquad k \in \mathbb{N}.$$

From the BVP for Y

$$\frac{Y''}{Y} = \mu - \lambda\,, Y(0) = Y(\pi) = 0\,. \Longrightarrow Y(y) = b_1 \sin(\sqrt{\lambda - \mu} y) + b_2 \cos(\sqrt{\lambda - \mu} y)$$

we obtain

$Y(y) = b_1 \sin(\sqrt{\lambda - \mu} y) + b_2 \cos(\sqrt{\lambda - \mu} y)$, $Y(0) = 0 \Longrightarrow b_2 = 0\,,$

$Y(\pi) = b_1 \sin(\sqrt{\lambda - \mu}\pi) = 0 \Longrightarrow \sqrt{\lambda - \mu} = n \in \mathbb{N}\,.$

With $\mu_k = k^2$ we have $\lambda_k = \mu_k + n^2 = k^2 + n^2$ and

$$Y_n(x) = b_n \sin(\sqrt{\lambda - \mu} y) = b_n \sin(ny), \qquad n \in \mathbb{N}.$$

What remains is the differential equation for T

$$\dot{T} = -\lambda T \Longrightarrow T(t) = c \cdot e^{-\lambda t}$$

or in terms of X_k, Y_k with $\lambda_k = k^2 + n^2$

$$T_k(t) = c_k\, e^{-(k^2+n^2)t}\,.$$

c) Following part a) and b) the functions

$$u_{nk} = c_k\, e^{-(n^2+k^2)t} a_k \sin(kx) b_n \sin(ny) = a_{kn} e^{-(n^2+k^2)t} \sin(kx)\sin(ny)$$

solve the differential equation and fulfill the boundary values. Because of the linearity of the problem, every linear combination of such solutions is again a solution. Transition to an infinite number of summands (without checking the convergence) results in the solution ansatz

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{kn} e^{-(n^2+k^2)t} \sin(kx)\sin(ny)$$

The initial value $u(x, y, 0) = \frac{1}{2}(\sin(2x) + \sin(x))\sin(y)$ for $x, y \in [0, \pi]$ yields $a_{11} = a_{21} = 1/2$ and $a_{kn} = 0$ otherwise. The solution reads as (can be confirmed by test substitution)

$$u(x, y, t) = \frac{1}{2} e^{-2t} \sin x \sin y + \frac{1}{2} e^{-5t} \sin(2x) \sin y\,.$$

It holds for $t \longrightarrow \infty$ because of the boundedness of the sine terms and $\lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty} e^{-5t} = 0$ it obviously tends to zero.