Fachbereich Mathematik der Universität Hamburg **SoSe 2022**
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Differential Equations II for Engineering Students

Work sheet 5

Exercise 1:

Using a suitable product ansatz, solve the following Dirichlet boundary value problem for the Laplace equation on the circle $r^2 = x^2 + y^2 \leq 9$.

$$\begin{aligned} r^2 u_{rr} + r u_r + u_{\varphi\varphi} &= 0 \qquad && 0 \leq r < 3 \\ u(3, \varphi) &= \cos^2(\varphi) && \varphi \in \mathbb{R}. \end{aligned}$$

Hints:

To solve Euler's equation
$r^2 \cdot g''(r) + ar \cdot g'(r) + b \cdot g(r) = 0$ use the ansatz $g(r) = r^k$.
It holds: $\cos^2(\varphi) = \frac{1}{2}(1 + \cos(2\varphi))$.

Solution sketch:

By inserting the product ansatz: $u = v(r) \cdot w(\varphi)$ into the Laplace equation in polar coordinates we obtain

$$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0$$

$$r^2 v'' w + r v' w + v w'' = 0 \implies \frac{r^2 v'' + r v'}{v} = -\frac{w''}{w} = \lambda$$

The solutions of $w''/w = -\lambda$ depend on the sign of λ, but only 2π–periodic solutions are possible here:

$$w_k(\varphi) = c_1 \cos(k\varphi) + c_2 \sin(k\varphi), \qquad \lambda = k^2 \quad k \in \mathbb{N}_0$$

For v we then get the (Euler's) differential equation

$$r^2 v'' + r v' - k^2 v = 0$$

For $k = 0$ we have the solution $v_0 = a_0 + b_0 \ln(r)$.

For $k \neq 0$ using the ansatz $v = r^m$ we have two solutions $v_k = r^k$ and $\tilde{v}_k = r^{-k}$.

Since the solution should be defined in a circle around zero, i.e. it should remain restricted, the negative powers and the $\ln-$ terms are not possible here. So we get the solution representation:

$$u(r, \varphi) = a_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\varphi) + d_k \sin(k\varphi)\right) r^k .$$

The boundary data gives us a condition

$$\begin{aligned} u(3,\varphi) &= a_0 + \sum_{k=1}^{\infty} \left(c_k \cos(k\varphi) + d_k \sin(k\varphi)\right) 3^k \\ &= \cos^2(\varphi) = \frac{1}{2} \left(1 + \cos(2\varphi)\right) . \end{aligned}$$

A coefficient comparison then yields the solution

$$u(r,\varphi) = \frac{1}{2} + \frac{r^2}{18} \cos(2\varphi) .$$

Exercise 2:

Determine the solution to the initial boundary value problem (IBVP)

$$\begin{aligned} u_t - u_{xx} &= \sin(x)\,t && 0 < x < \pi\,,\ 0 < t\,,\\ u(x,0) &= 4\sin(3x) + \tfrac{x}{\pi} && 0 \le x \le \pi,\\ u(0,t) &= \phi_1(t) = 0 && 0 \le t,\\ u(\pi,t) &= \phi_2(t) = 1 && 0 \le t. \end{aligned}$$

Note: First homogenize the boundary conditions by using the function

$$v(x,t) = u(x,t) - \phi_1(t) - \frac{x-a}{b-a}\left(\phi_2(t) - \phi_1(t)\right)$$

with $a = 0$ and $b = \pi$ and then replace the u–expressions with corresponding v–expressions. You get e.g.

$$u_t = v_t + \dot{\phi}_1 + \frac{x-a}{b-a}\left(\dot{\phi}_2 - \dot{\phi}_1\right).$$

Solution:

Step 1: Homogenization of the boundary conditions

$$v(x,t) = u(x,t) - \varphi_1(t) - \frac{x-0}{\pi-0}(\varphi_2(t) - \varphi_1(t)) = u(x,t) - \frac{x}{\pi}$$

New IBVP:

$$\begin{aligned} v_t - v_{xx} &= \sin(x)\,t\\ v(x,0) &= u(x,0) - \frac{x}{\pi} = 4\sin(3x)\\ v(0,t) &= v(\pi,t) = 0 \end{aligned}$$

Step 2: Solving the homogeneous IBVP

With $\omega = \frac{\pi}{L} = 1$ and $c = 1$ we make an ansatz

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\omega x)\,.$$

Every function $a_n(t)\sin(n\omega x)$ fulfills the homogeneous boundary conditions. Thus, every sum of these functions also satisfies the homogeneous boundary conditions.

Inserting the ansatz into the differential equation and Fourier expansion of the odd, $2L-$ periodic continuations of the right-hand side yield the ordinary differential equations

$$\dot{a}_n(t) + cn^2\omega^2 a_n(t) = c_n(t),\ a_n(0) = b_n$$

b_n : Fourier coefficients of the odd $2L-$ periodic continuation of $v(x,0)$:

$$4\sin(3x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Hence $b_n = 0,\ \forall n \neq 3, \qquad b_3 = 4\,.$

$c_n(t)$: Fourier coefficients of the odd, $2L-$ periodic continuation of $h(x,t)$:

$$\sin(x)\,t = \sum_{n=1}^{\infty} c_n(t)\sin(nx)$$

Hence $c_n(t) \equiv 0, \forall n \neq 1, \qquad c_1(t) = t\,.$

For $n \notin \{1,3\}$ we obtain the initial value problem

$\dot{a}_n(t) + n^2 a_n(t) = c_n(t) = 0,\; a_n(0) = b_n = 0$

with the solution $a_n(t) \equiv 0\,.$

For $n = 1:$

$\dot{a}_1(t) + 1^2 a_1(t) = c_1(t) = t,\; a_1(0) = b_1 = 0$

The associated homogeneous differential equation

$\dot{a}_{1h}(t) + a_{1h}(t) = 0$ is obviously solved by $e^{-t}\,.$

Using the ansatz

$a_{1p}(t) = \alpha + \beta t$ we obtain the particular solution of the non-homogeneous problem for a_1

$a_{1p}(t) = t - 1\,.$ Hence we have

$a_1(t) = \gamma e^{-t} + t - 1 \qquad \text{and} \qquad a_1(0) = \gamma e^0 + 0 - 1 = 0 \Longrightarrow \gamma = 1$

$\Longrightarrow \quad \boxed{a_1(t) = e^{-t} + t - 1}$

For $n = 3:$

$\dot{a}_3(t) + 3^2 a_3(t) = c_3(t) = 0,\; a_3(0) = b_3 = 4$

$\Longrightarrow a_3(t) = \tilde{\gamma} e^{-9t} \qquad \text{and} \qquad a_3(0) = \tilde{\gamma} e^0 = 4$

Hence $\boxed{a_3(t) = 4e^{-9t}}$

$$\begin{aligned} v(x,t) &= \sum_{k=1}^{\infty} a_n(t)\sin(n\omega x) = a_1(t)\sin(x) + a_3(t)\sin(3x) \\ &= \big(e^{-t} + t - 1\big)\sin(x) + 4e^{-9t}\sin(3x) \end{aligned}$$

Step 3: Compose the solution

The solution to the original problem is

$$u(x,t) = v(x,t) + \frac{x}{\pi} = 4e^{-9t}\sin(3x) + (e^{-t} + t - 1)\sin x + \frac{x}{\pi}.$$