Fachbereich Mathematik der Universität Hamburg **SoSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 5

Exercise 1:

a) Derive Green's function for a sphere of radius R around the origin.

b) Let

$$\begin{aligned} \bar{x} &= \text{reflection of } x = (x_1, x_2) \text{ through the } x_1\text{–axis}, \\ \hat{x} &= \text{reflection of } x \text{ through the } x_2\text{–axis}, \\ \tilde{x} &= \text{reflection of } x \text{ through the origin} \end{aligned}$$

Show that the Green's function for the quadrant $x_1 > 0$, $x_2 > 0$ can be constructed using the correction function

$$\Phi^x(y) := \Phi(y - \bar{x}) + \Phi(y - \hat{x}) - \Phi(y - \tilde{x})$$

Solution hints for the exercise 1

a) To construct a correction function, we choose the reflection on the sphere:

$$\tilde{x} := R^2 \frac{x}{\|x\|^2}$$

Then it holds for y from the edge of the sphere, i.e. $\|y\| = R$

$$\begin{aligned} \|y - \tilde{x}\|^2 &= (y - \tilde{x})^T (y - \tilde{x}) = R^2 - 2y^T \tilde{x} + \|\tilde{x}\|^2 \\ &= R^2 - 2\frac{R^2}{\|x\|^2} y^T x + \frac{R^4}{\|x\|^4} x^T x = \frac{R^2}{\|x\|^2} \left(\|x\|^2 - 2y^T x + R^2\right) \\ &= \frac{R^2}{\|x\|^2} \left(\|x\|^2 - 2y^T x + y^T y\right) = \frac{R^2}{\|x\|^2} \|y - x\|^2 \end{aligned}$$

Hence it holds $\|y - x\| = \dfrac{\|x\|}{R} \|y - \tilde{x}\| \quad \forall y : \|y\| = R$ and with

$$\Phi^x(y) := \Phi\left(\frac{\|x\|}{R}(y - \tilde{x})\right)$$

we obtain

$$\begin{aligned} \Delta\Phi^x &= 0 && \forall x, y : \|y\|,\ \|x\| < R \\ \Phi^x(y) &= \Phi(y-x) && \forall y : \|y\| = R. \\ G(x,y) &:= \Phi(y-x) - \Phi^x(y) \end{aligned}$$

b) Certainly, it holds $\Delta_y \Phi^x = 0$ on $y_1, y_2 > 0,\ x \in \mathbb{R}^2$. It remains to show that the values of $\Phi(y-x)$ and $\Phi^x(y)$ coincide on the boundary. It holds

$$\begin{aligned} \phi^x(y) &= \Phi(y-\bar{x}) + \Phi(y-\hat{x}) - \Phi(y-\tilde{x}) \\ &= \Phi(y_1 - x_1, y_2 + x_2) + \Phi(y_1 + x_1, y_2 - x_2) - \Phi(y_1 + x_1, y_2 + x_2). \end{aligned}$$

For $y_2 = 0,\ y_1 > 0$ we have

$$\Phi^x(y) = \Phi(y_1 - x_1, x_2) + \Phi(y_1 + x_1, -x_2) - \Phi(y_1 + x_1, x_2).$$

Since Φ only depends on the absolute values of the components, the last two terms cancel each other out and we obtain

$$\begin{aligned} \Phi^x(y) =& \Phi(y_1 - x_1, x_2) = \Phi(y_1 - x_1, -x_2) \\ =& \Phi(y_1 - x_1, y_2 - x_2) = \Phi(y - x). \end{aligned}$$

The argument for $y_1 = 0,\ y_2 > 0$ is analogous:

$$\begin{aligned} \Phi^x(y) &= \Phi(-x_1, y_2 + x_2) + \Phi(x_1, y_2 - x_2) - \Phi(x_1, y_2 + x_2) \\ &= \Phi(x_1, y_2 - x_2) = \Phi(0 - x_1, y_2 - x_2) = \Phi(y - x). \end{aligned}$$

For $y_1 = y_2 = 0$ we obtain

$$\begin{aligned} \Phi^x(y) &= \Phi(-x_1, x_2) + \Phi(x_1, -x_2) - \Phi(x_1, x_2) \\ &= \Phi(-x_1, -x_2) = \Phi(0 - x_1, 0 - x_2) = \Phi(y - x). \end{aligned}$$

Exercise 2

Determine the solutions to the following tasks using the suitable product ansatz.

a)
$$u_t = u_{xx} \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$
$$u(x,0) = \sin(x) + 2\cos(2x) \qquad x \in \mathbb{R}\cdot$$

b)
$$\begin{aligned} u_t - u_{xx} &= 0 \qquad 0 < x < \pi,\ t \in \mathbb{R}^+, \\ u(x,0) &= \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4} \qquad 0 < x < \pi \\ u(\pi,t) &= u(0,t) = 0 \qquad t > 0. \end{aligned}$$

Solution hints to Exercise 2:

a)
$$u_t = u_{xx} \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$
$$u(x,0) = \sin(x) + 2\cos(2x) \qquad x \in \mathbb{R}\cdot$$

The ansatz $u(x,t) = X(x)T(t)$ (see notes on lecture 7) gives us

$$X\dot{T} = T \cdot X''$$
$$\frac{X''}{X} = \frac{\dot{T}}{T} = \lambda = \text{constant}$$
$$T(t) = T(0)e^{\lambda t}$$
$$X(x) = a\cos(\sqrt{-\lambda}x) + b\sin(\sqrt{-\lambda}x)$$

We solve the problem for the right-hand side $\sin(x)$. It must hold

$$u_1(x,0) = T_1(0)\left(a_1\cos(\sqrt{-\lambda_1}x) + b_1\sin(\sqrt{-\lambda_1}x)\right) = \sin(x)$$

We have a solution as $a_1 = 0, b_1 = 1/T_1(0)$, and $\lambda_1 = -1$. Hence we obtain

$$u_1(x,0) = \sin(x)e^{-t}$$

We proceed analogously with $2\cos(2x)$ and obtain $a_2 = 2/T_2(0),\ b_2 = 0,\ \lambda_2 = -4$ and hence

$$u_2(x,t) = 2\cos(2x)e^{-4t}$$

We conclude with the superposition principle:

$$u(x,t) = u_1(x,t) + u_2(x,t) = \sin(x)e^{-t} + 2\cos(2x)e^{-4t}$$

b) Product ansatz for

$$\begin{aligned} u_t - u_{xx} &= 0 \qquad 0 < x < \pi,\ t \in \mathbb{R}^+, \\ u(x,0) &= \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4} \qquad 0 < x < \pi \\ u(\pi,t) = u(0,t) &= 0 \qquad t > 0. \end{aligned}$$

gives us as in part a)

$$T(t) = T(0)e^{\lambda t}$$

$$X(x) = a\cos(\sqrt{-\lambda}x) + b\sin(\sqrt{-\lambda}x)$$

It follows from the boundary data that the cosine terms must vanish ($u(0,t) = 0$), and

$$\sin(\sqrt{-\lambda}\pi) = 0 \implies \lambda_k = -k^2 \qquad k \in \mathbb{Z}\,.$$

Every function of the form

$$u_k(x,t) = c_k e^{-k^2 t} sin(kx)$$

solves the differential equation and fulfills the boundary data. We make an ansatz

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2 t}\sin(kx)$$

The initial condition remains to be fulfilled. To do this, we expand the right-hand side of the initial condition into a Fourier sine series in terms of the functions $\sin(kx)$. This can be read directly here because of the form of the right side. It must hold

$$u(x,0) = \sum_{k=1}^{\infty} c_k e^{0\cdot t}\sin(kx) = \sum_{k=1}^{\infty} c_k \sin(kx) = \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4}$$

and hence we have

$$c_2 = \frac{1}{2}, \qquad c_4 = \frac{1}{4}, \qquad c_k = 0 \quad \text{sonst}$$

So the solution is

$$u(x,t) = \frac{1}{2}e^{-4t}\sin(2x) + \frac{1}{4}e^{-16t}\sin(4x).$$