Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 4

Exercise 1:

Determine the type of the following differential equations

a) $u_{xx} + 4u_{xt} - 5u_{tt} = 0,$

b) $10u_{xx} + 6u_{xy} + u_{yy} = 0$

c) $4x^2\, u_{xx} + 8xy\, u_{xy} + y^2\, u_{yy} + 2x\, u_x = 0$

Solution:

a) $u_{xx} - 4u_{xt} - 5u_{tt} = 0$

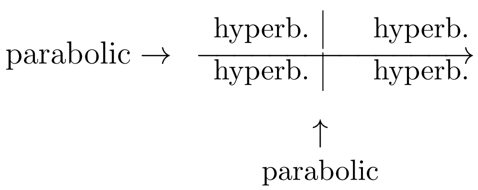
$$1 \cdot (-5) - (-2)^2 < 0 \qquad \text{hyperbolic}$$

b) $10u_{xx} + 6u_{xy} + u_{yy} = 0$

$$10 - 3^2 = 1 \qquad \text{elliptic}$$

c) $4x^2\, u_{xx} + 8xy\, u_{xy} + y^2\, u_{yy} + 2x\, u_x = 0$

$$4x^2 \cdot y^2 - 16x^2y^2 = -12x^2y^2 \Longrightarrow \begin{cases} \text{parabolic for} & xy = 0 \\ \text{hyperbolic for} & xy \neq 0 \end{cases}$$



Exercise 2:
Determine all rotationally symmetrical solutions of the following boundary value problem

$$\Delta u = -\frac{1}{\sqrt{x^2+y^2}} \quad \text{for } 1 < x^2+y^2 < 9,$$
$$u(x,y) = 1 \quad \text{on } x^2+y^2 = 1,$$
$$u(x,y) = 2 \quad \text{on } x^2+y^2 = 9.$$

Note: Laplace equation in polar coordinates is given by $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = 0$ *.*
Rotationally symmetrical means that it does not depend on ϕ

Solution:

As in the lecture, for rotationally symmetric problems with $v(r) := u(x(r), y(r))$, $w := v'$ and right-hand size $-f(r)$:

$$v'' + \frac{2-1}{r}v' = -f(r) = -r^{-1}, \Longrightarrow w' + \frac{1}{r}w = -r^{-1} \Longrightarrow w_h = \alpha/r,$$
$$w_p = \alpha(r)/r\,, \Longrightarrow \alpha'(r)/r = -1/r \Longrightarrow \alpha'(r) = -1 \Longrightarrow \alpha(r) = -r + c$$
$$\Longrightarrow w(r) = -1 + \alpha/r \Longrightarrow v(r) = \alpha\ln(r) - r + \beta$$
$$\Longrightarrow u(x,y) = \alpha\ln\left(\sqrt{x^2+y^2}\right) - \sqrt{x^2+y^2} + \beta$$

The boundary data gives us for $x^2+y^2 = 1$:

$$u(x,y) = -1 + \beta = 1 \Longrightarrow \beta = 2$$

and for $x^2+y^2 = 9$

$$u(x,y) = \alpha\ln(3) - 3 + 2 = 2 \Longrightarrow \alpha = 3/\ln(3)$$

Exercise 3: Only for very fast students

a) Let u be the solution of the boundary value problem

$$\begin{aligned} \Delta u &= -1 \qquad & |x| < 1, |y| < 1, \\ u(x,y) &= \ \ 0 & |x| = 1 \text{ or } |y| = 1 \end{aligned}$$

and $v(x,y) = u(x,y) + \frac{1}{4}(x^2 + y^2)$.

Show that $v(x,y)$ solves Laplace equation, and determine the upper and lower bounds for $u(0,0)$.

b) Let $u(x,y)$ is a solution to the problem:

$$\begin{aligned} \Delta u &= 0, \qquad \text{in } \Omega :=]0,2[\times]0,1[\\ u(x,y) &= 3x^2 \qquad\qquad \text{on } \partial\Omega . \end{aligned}$$

Determine, without computing u, for each of the following statements whether it is true. Explain your answers.

- It holds $\max_{(x,y)\in\bar{\Omega}} u(x,y) = 2$.
- It holds $\min_{(x,y)\in\bar{\Omega}} u(x,y) = 0$.
- $u(x,y) = 3x^2 - 3y^2$ is a solution to the boundary value problem.

Solution:

a) It holds

$$\begin{aligned} v_{xx} + v_{yy} &= u_{xx} + \left(\frac{1}{4}(x^2+y^2)\right)_{xx} + u_{yy} + \left(\frac{1}{4}(x^2+y^2)\right)_{yy} \\ &= \Delta u + \frac{1}{4}(x^2)_{xx} + \frac{1}{4}(y^2)_{yy} = -1 + \frac{1}{2} + \frac{1}{2} = 0 . \end{aligned}$$

So the function v fulfills Laplace equation. It attains its minimum and maximum on the boundary of rectangle $[-1,1] \times [-1,1]$. There it holds

$v(x,y) = u(x,y) + \frac{1}{4}(x^2+y^2) \leq 0 + \frac{1}{4}(1^2+1^2) = \frac{1}{2}$.

and

$v(x,y) \geq 0 + \frac{1}{4}$ (it is always $|x|$ or $|y| = 1$.)

So in the entire rectangle it holds $\frac{1}{4} \leq v(x,y) \leq \frac{1}{2}$.

Hence we obtain $u(0,0) = v(0,0) \in [\frac{1}{4}, \frac{1}{2}]$.

b) The first statement is false because, for example, $u(1,0) = 3 > 2$.

The second statement follows from the maximum principle.

$u(x,y) = 3x^2 - 3y^2$ is a solution to the differential equation, but it does not satisfy the boundary conditions.