Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 4

Exercise 1: Given a differential equation

$$2u_{xx} + 4u_{xy} + 2u_{yy} + \sqrt{2}(u_x + u_y) \;=\; 0\,.$$

a) Determine the type of the equation.

b) Transform the equation to its normal form.

c) Determine the general solution of the transformed differential equation and perform the backwards transformation.

Solution:

a) $2 \cdot 2 - 2^2 = 0$. It is a parabolic differential equation.

b) In the matrix form:

$$(\nabla^T A \nabla)u + (b^T \nabla)u = 0, \qquad A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

Characteristic polynomial: $(2-\lambda)^2 - 4 = 0 \implies \lambda_1 = 0,\ \lambda_2 = 4\,.$

The eigenvector $\boldsymbol{w}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for eigenvalue 0 can be obtained straightforwardly.

The eigenvector $\boldsymbol{w}_2$ for eigenvalue 4 is orthogonal to $\boldsymbol{w}_1$. So $\boldsymbol{w}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

So we obtain the transformation:

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \eta \\ \tau \end{pmatrix} = S^T \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \end{pmatrix}$$

We define $\tilde{u}(\eta, \tau) = \tilde{u}(\eta(x,y), \tau(x,y)) = u(x,y)$. It holds:

$$\nabla_{xy} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = S \cdot \nabla_{\eta\tau} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \tau} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau} \\ -\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \tau} \end{pmatrix}$$

so $u_x + u_y = \sqrt{2}\tilde{u}_\tau$

So we obtain a new differential equation

$$4\tilde{u}_{\tau\tau} + 2\tilde{u}_\tau = 0 \iff \tilde{u}_{\tau\tau} + \frac{1}{2}\tilde{u}_\tau = 0$$

c) To determine the general solution we define $z(\eta,\tau)=\tilde{u}_\tau(\eta,\tau)$. Thus, we obtain
$z_\tau = -\frac{1}{2}z \iff \dfrac{dz}{d\tau} = -\dfrac{z}{2} \iff \dfrac{dz}{z} = -\dfrac{d\tau}{2} \iff \ln|z| = -\dfrac{\tau}{2} + k(\eta) \iff$
$z = K(\eta)\cdot e^{-\dfrac{\tau}{2}}$
Integration w.r.t τ gives (with $f(\eta) = -2K(\eta)$
$\tilde{u}(\eta,\tau) = f(\eta)e^{-\dfrac{\tau}{2}} + g(\eta)$
or
$u(x,y) = f\left(\dfrac{x-y}{\sqrt{2}}\right)e^{-\frac{x+y}{2\sqrt{2}}} + g\left(\dfrac{x-y}{\sqrt{2}}\right).$

Alternative solution:

$4\tilde{u}_{\tau\tau} + 2\tilde{u}_\tau = 0 \iff \tilde{u}_{\tau\tau} + \frac{1}{2}\tilde{u}_\tau = 0$
Characteristic polynomial:
$\lambda^2 + \frac{1}{2}\lambda = 0 \Longrightarrow \lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}$

General solution:
$\tilde{u}(\eta,\tau) = c_2(\eta) + c_1(\eta)e^{-\dfrac{\tau}{2}}$

$u(x,y) = c_1\left(\dfrac{x-y}{\sqrt{2}}\right) + c_2\left(\dfrac{x-y}{\sqrt{2}}\right)e^{-\frac{x+y}{2\sqrt{2}}}.$

Exercise 2:

Given the initial value problem

$$\begin{aligned} u_{tt} + u_{xt} - 2u_{xx} &= 0 \quad \text{for } x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= \cos(x) \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= -4\sin(x). \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Solve the problem using the substitution $\alpha = x + t,\ \mu = x - 2t\,.$

Note: The procedure is analogous to the derivation of the solution to the Cauchy problem for the wave equation from the lecture. Alternatively: convert the derivatives in terms of x, t into derivatives in terms of α, μ.

Solution:

Using the substitution $\alpha = x + t\,,\ \mu = x - 2t$ we obtain

$$x = \frac{1}{3}(2\alpha + \mu), \quad t = \frac{1}{3}(\alpha - \mu),$$

so

$$u(x,t) = u\left(\frac{1}{3}(2\alpha + \mu), \frac{1}{3}(\alpha - \mu)\right) =: v(\alpha, \mu)$$

Further, it holds

$$\begin{aligned} v_\alpha &= \frac{2}{3}u_x + \frac{1}{3}u_t \\ v_{\alpha\mu} &= \frac{2}{3}\cdot\frac{1}{3}u_{xx} + \frac{2}{3}\cdot\left(-\frac{1}{3}\right)u_{xt} + \frac{1}{3}\cdot\frac{1}{3}u_{tx} + \frac{1}{3}\cdot\left(-\frac{1}{3}\right)u_{tt} \\ &= -\frac{1}{9}\left(u_{tt} + u_{xt} - 2u_{xx}\right) = 0 \end{aligned}$$

Hence it follows

$$v(\alpha, \mu) = \phi(\alpha) + \chi(\mu)$$

and

$$u(x,t) = v(\alpha, \mu) = \phi(x+t) + \chi(x - 2t)$$

with sufficiently smooth functions ϕ and χ. From the initial values we obtain two conditions

$$\begin{aligned} u(x,0) &= \phi(x) + \chi(x) \overset{!}{=} \cos(x) \quad \text{as well as} \\ u_t(x,0) &= \phi'(x) - 2\chi'(x) \overset{!}{=} -4\sin(x) \;\Rightarrow\; \phi(x) - 2\chi(x) \overset{!}{=} -4\int_{x_0}^{x} \sin(z)\,dz. \end{aligned}$$

Adding twice the first equation to the second equation, we get

$$3\phi(x) = 2\cos(x) + 4\cos(x) - 4\cos(x_0).$$

Subtracting the two conditions leads to

$$3\chi(x) = \cos(x) - 4\cos(x) + 4\cos(x_0).$$

The solution to the initial value problem is therefore given by

$$u(x,t) = 2\cos(x+t) - \cos(x-2t).$$

Alternative solution:

Using the substitution $\alpha = x+t,\ \mu = x-2t,\ v(\alpha,\mu) = u(x,t)$ we obtain

$$\begin{aligned}
u_x &= v_\alpha \cdot \alpha_x + v_\mu \cdot \mu_x = v_\alpha + v_\mu \\
u_t &= v_\alpha \cdot \alpha_t + v_\mu \cdot \mu_t = v_\alpha - 2v_\mu \\
u_{xx} &= v_{\alpha\alpha} \cdot \alpha_x + v_{\alpha\mu} \cdot \mu_x + v_{\mu\alpha} \cdot \alpha_x + v_{\mu\mu} \cdot \mu_x = v_{\alpha\alpha} + 2v_{\alpha\mu} + v_{\mu\mu} \\
u_{xt} &= v_{\alpha\alpha} \cdot \alpha_t + v_{\alpha\mu} \cdot \mu_t + v_{\mu\alpha} \cdot \alpha_t + v_{\mu\mu} \cdot \mu_t = v_{\alpha\alpha} - v_{\alpha\mu} - 2v_{\mu\mu} \\
u_{tt} &= v_{\alpha\alpha} \cdot \alpha_t + v_{\alpha\mu} \cdot \mu_t - 2v_{\mu\alpha} \cdot \alpha_t - 2v_{\mu\mu} \cdot \mu_t = v_{\alpha\alpha} - 4v_{\alpha\mu} + 4v_{\mu\mu}
\end{aligned}$$

$$u_{tt} + u_{xt} - 2u_{xx} = -9v_{\alpha\mu} = 0 \iff v_{\alpha\mu} = 0$$

Hence we obtain

$$v_\alpha = \phi(\alpha) \implies v(\alpha,\mu) = \Phi(\alpha) + \Psi(\mu) \implies u(x,t) = \Phi(x+t) + \Psi(x-2t)$$

Initial conditions:

$$u(x,0) = \Phi(x) + \Psi(x) = \cos(x), \qquad u_t(x,0) = \Phi'(x) - 2\Psi'(x) = -4\sin(x)$$

Differentiating the first equation and subtracting the second equation gives

$$\Phi'(x) + \Psi'(x) = -\sin(x) \longrightarrow \Psi'(x) = \sin(x)\,.$$

Substituting into the first equation of the last line gives

$$\Phi'(x) = -2\sin(x) \implies \Phi(x) = 2\cos(x) + C$$

$$u(x,t) = 2\cos(x+t) - \cos(x-2t)\,.$$

For $u_{tt} + (a+b)u_{tx} + abu_{xx}$ **analogous with substitution** $\alpha = x - bt,\ \mu = x - at$**.**

Exercise 3:

a) For which real values of α and for which real-valued functions $g:\mathbb{R}\to\mathbb{R}$ are the following functions harmonic in $\mathbb{R}^2$?

i) $\tilde{u}(x,y) = \cos(\alpha x)\cdot e^{3y}$, ii) $\hat{u}(x,y) = \sin(\alpha x)\cdot\cosh(3y)$

iii) $u(x,y) = \frac{1}{2}\cdot(x^3 + g(x)\cdot y^2)$.

b) Let $\Omega := \left\{(x,y)^T \in \mathbb{R}^2 : x^2+y^2 < 16\right\}$ and u be the solution of boundary value problem

$$\Delta u(x,y) = 0 \qquad \text{in } \Omega, \qquad u(x,y) = \frac{2y^2}{x^2+y^2} \qquad \text{on} \quad \partial\Omega.$$

Determine the value of u in the origin.

Note: $\sin^2(\varphi) = \dfrac{1-\cos(2\varphi)}{2}$.

Solution:

a) i) $\Delta\tilde{u}(x,y) = -\alpha^2\cos(\alpha x)\cdot e^{3y} + 9\cdot\cos(\alpha x)\cdot e^{3y}$.

$\tilde{u}$ is for $\alpha = \pm 3$ harmonic.

ii) $\Delta\hat{u}(x,y) = -\alpha^2\sin(\alpha x)\cdot\cosh(3y) + 9\sin(\alpha x)\cdot\cosh(3y)$.

$\hat{u}$ is also for $\alpha = \pm 3$ harmonic. Additionally there is a trivial solution for $\alpha = 0$.

iii) $\Delta u(x,y) = \frac{1}{2}\cdot(6x + g''(x)y^2 + 2g(x)) = 0 \qquad \forall y \in \mathbb{R}^2 \Longrightarrow g(x) = -3x$.

b) Let K_4 be the edge of the disk with radius 4 around zero and

$c(t) = (4\cos(t),\, 4\sin(t)), \qquad t\in[0,2\pi]$

a parametrization of K_4. Then it holds because of the mean value property

$$u(0,0) = \frac{1}{2\pi\cdot 4}\int_{K_4}\frac{2y^2}{x^2+y^2}\,d(x,y) = \frac{1}{8\pi}\int_0^{2\pi}\frac{2\cdot 16\sin^2(t)}{16\cos^2(t)+16\sin^2(t)}\cdot\|\dot{c}(t)\|\,dt$$

$$= \frac{1}{8\pi}\int_0^{2\pi}(1-\cos(2t))\cdot 4\,dt = 1.$$