Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
Prof. Dr. J. Behrens
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Work sheet 2

Exercise 1:

Determine the entropy solution to Burgers' equation $u_t + uu_x = 0$ with the initial data

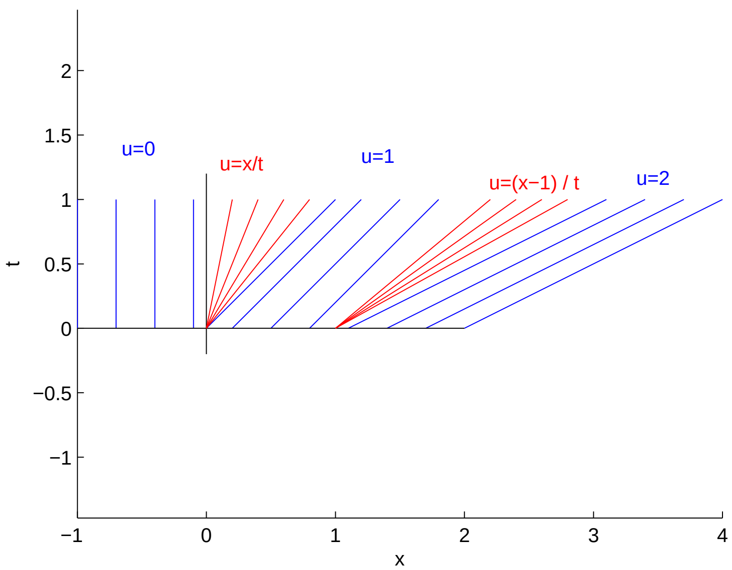
a) $u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 2 & 1 < x \end{cases}$ and **b)** $u(x,0) = \begin{cases} 2 & x < 0 \\ 1 & 0 \le x \le 2 \\ 0 & 2 < x \end{cases}$

Solution sketch:

a) $u_t + uu_x = 0$

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 2 & x > 1 \end{cases}$$

$\longrightarrow$ two rarefaction waves



$$u(x,t)=\begin{cases}0 & x\leq 0\\ \frac{x}{t} & 0\leq x\leq t\\ 1 & t\leq x\leq t+1\\ \frac{x-1}{t} & t+1\leq x\leq 2t+1\\ 2 & x\geq 2t+1\end{cases}$$

b) $u_t + uu_x = 0$

$$u(x,0)=\begin{cases}2 & x<0\\ 1 & 0\leq x\leq 2\\ 0 & x>2\end{cases}$$

2 shock waves:

$s_1(t)$ with $s_1(0)=0$ $\qquad \dot{s}_1(t)=\dfrac{2+1}{2}=\dfrac{3}{2}$

$$\boxed{s_1(t)=\frac{3}{2}t}$$

$s_2(t)$ with $s_2(0)=2$ $\qquad \dot{s}_2(t)=\dfrac{0+1}{2}=1/2$

$$\boxed{s_2(t)=2+\frac{t}{2}}$$

for $t=2$ is $s_1(t)=s_2(t)$

$\longrightarrow$ new shock wave

$s_3(t)$ with $s_3(2)=s_1(2)=s_2(2)=3$ $\qquad \dot{s}_3(t)=\dfrac{2+0}{2}=1$ so

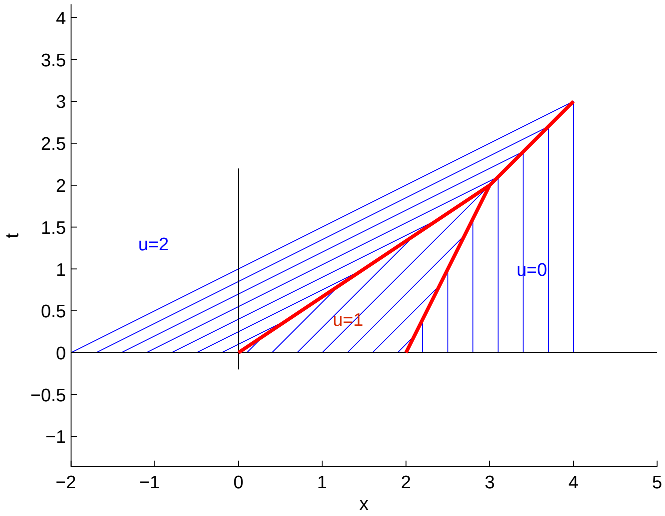
$$\boxed{s_3(t)=t+1}$$

for $t<2$

$$u(x,t)=\begin{cases}2 & x<\frac{3}{2}t\\ 1 & \frac{3}{2}t\leq x\leq 2+\frac{t}{2}\\ 0 & x>2+\frac{t}{2}\end{cases}$$

for $t\geq 2$

$$u(x,t)=\begin{cases}2 & x\leq t+1\\ 0 & x>t+1\end{cases}$$


u=2
u=1
u=0
t
x
4
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−2
−1
0
1
2
3
4
5

Exercise 2)

Given a conservation equation $u_t + \left(\frac{u^4}{16}\right)_x = 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+ .$

a) Are the characteristics $(x(t), t)$ straight lines? Explain your answer.

b) Given the initial data $u(x,0) = 2 + \arctan(x)$, $x \in \mathbb{R}$. Determine the characteristic through the point $(0,0)$.

c) Check which of the functions $u^*, \tilde{u}, \hat{u}$ given below is the (weak) entropy solution for the initial values

$$u(x,0) = \begin{cases} 2 & \text{for } x \le 0, \\ 1 & \text{for } x > 0 \end{cases}$$

$$u^*(x,t) = \begin{cases} 2 & \text{for } x \le \frac{3}{2}t, \\ 1 & \text{for } x > \frac{3}{2}t . \end{cases} \qquad \tilde{u}(x,t) = \begin{cases} 2 & \text{for } x \le \frac{15}{16}t, \\ 1 & \text{for } x > \frac{15}{16}t . \end{cases}$$

$$\hat{u}(x,t) = \begin{cases} 2 & \text{for } x \le 0, \\ 2 - \frac{x}{t} & \text{for } 0 < x \le t, \\ 1 & \text{for } x > t . \end{cases}$$

Solution sketch:

a) For the characteristics, on one hand, we have

$\frac{du}{dt} = 0 \Longrightarrow u$ is constant along the characteristics.

On the other hand, the characteristics have slope $\frac{dx}{dt} = \frac{u^3}{4}$.

So the slope of the characteristics is constant. Hence they are straight lines.

b) The characteristic through the point $(0,0)$ is a straight line $(x(t), t)$ through $(0,0)$ with

$\dot{x}(t) = \frac{u(0,0)^3}{4} = \dfrac{(2 + \arctan(0))^3}{4} = 2$, so the line $x = 2t$.

c) The shock front $s(t)$ of the entropy solution must fulfill the jump condition $\dot{s}(t) = \dfrac{f(u_l) - f(u_r)}{u_l - u_r}$. So we have

$$\dot{s}(t) = \frac{\frac{u_l^4}{16} - \frac{u_r^4}{16}}{u_l - u_r} = \frac{\frac{2^4}{16} - \frac{1^4}{16}}{2 - 1} = \frac{15}{16} .$$

u^* does not satisfy the jump condition: $\dot{s}(t) = \frac{15}{16}$.

$\tilde{u}$ satisfies the jump condition and is a classical solution outside the shock front, i.e. is overall a weak entropy solution.

$\hat{u}$ has no discontinuities. There is no jump condition to be fulfill! But for $0 < x < t$ it is not a solution to the differential equation.

It holds $\hat{u}_t + \frac{\hat{u}^3}{4}\,\hat{u}_x = \frac{x}{t^2} + (2 - \frac{x}{t})^3(\frac{-1}{4t}) = 0$

$$\implies \frac{4xt^2 - (2t-x)^3}{4t^4} = 0 \iff x^3 - 6tx^2 + 16xt^2 - 8t^3 = 0$$

For fixed t this condition can be fulfilled for at most three different x, but certainly not for all $0 < x \leq t$.