Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
Prof. Dr. J. Behrens
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Homework sheet 3

Exercise 1: Determine the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

at the time $t = 2$. What new problem occurs at $t = 2$?

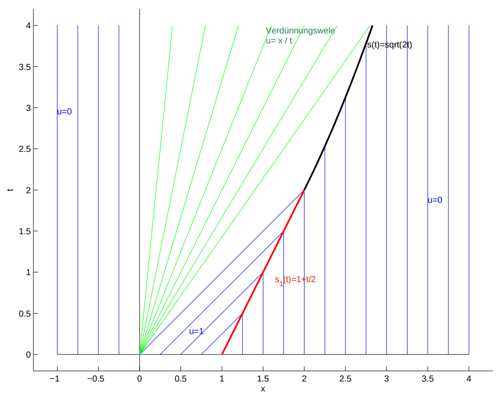
Alternatively: Determine the solution for $t > 2$.

Solution sketch: $u_t + uu_x = 0$

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

It is clear that

- the solution is constant along the characteristic lines
- the characteristics are straight lines with the slope $1/u_0$ in the $(x,t)-$ plane



So first of all, we get

$\dot{s}(t) = \dfrac{1+0}{2}$, $s(t) = 1 + \frac{t}{2}$:

$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \frac{x}{t} & 0 \le x \le t \\ 1 & t \le x \le 1 + \frac{t}{2} \\ 0 & x > 1 + \frac{t}{2}. \end{cases}$$

This solution is valid until t^* with $t^* = 1 + \frac{t^*}{2}$, i.e. $t^* = 2$. At time $t^* = 2$ the rarefaction wave meets the shock wave. For $t \ge 2$ holds for the discontinuity with

$u_l = \dfrac{x}{t}, \qquad u_r = 0$ and $x = s(t)$ on the discontinuity curve

$\dot{s}(t) = \dfrac{\frac{s(t)}{t} + 0}{2} = \dfrac{s(t)}{2t}$ This is an ordinary differential equation for $s(t)$.

$$\left.\begin{array}{l} \dfrac{ds}{s} = \dfrac{dt}{2t} \qquad \Longrightarrow \quad s = c\sqrt{t} \\ s(2) = 2 \end{array}\right\} \quad \Longrightarrow \qquad c = \sqrt{2}, \;\; s = \sqrt{2t}$$

So the discontinuity moves on the curve $x(t) = \sqrt{2t}$.

$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \frac{x}{t} & 0 \le x \le \sqrt{2t} \\ 0 & x > \sqrt{2t}. \end{cases}$$

Exercise 2:

Determine entropy solutions to the differential equation

$$u_t + (f(u))_x = 0$$

with the flow function $f(u) = \frac{(u-2)^4}{2}$ and initial conditions

a) $u(x,0) = \begin{cases} 2 & x \le 0, \\ 1 & 0 < x, \end{cases}$ and **b)** $u(x,0) = \begin{cases} 1 & x \le 0, \\ 2 & 0 < x. \end{cases}$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

Solution:

Using usual notation we have $f(u) = \frac{(u-2)^4}{2}$. **[1 point]**

On the characteristic curves it holds

$\dot{x}(t) = f'(u) = 2(u-2)^3$ and $\dot{u}(t) = 0$.

The characteristics are straight lines with the constant slope $2(u(x(0),0)-2)^3$.

In part a), an ambiguity of the solution obtained using the methods of characteristics arises immediately (i.e. already at $t = 0$). A shock front $s(t)$ must be introduced with $u_l = 2$ and $u_r = 1$ **[1 point]**

with

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{(2-2)^4}{2} - \frac{(1-2)^4}{2}}{2-1} = -\tfrac{1}{2} \quad \textbf{[1 point]}$$

We obtain

$$u(x,t) = \begin{cases} u_l = 2 & x < s(t) = -\frac{t}{2} \\ u_r = 1 & -\frac{t}{2} < x. \quad \textbf{[1 point]} \end{cases}$$

For part b) the method of characteristics gives

$$u(x,t) = \begin{cases} 1 & x \le x_0 + f'(u_l)t = 0 + 2(1-2)^3 t = -2t\,, \\ ? & -2t \le x \le 0\,, \\ 2 & x \ge x_0 + f'(u_r)t = 0 + 2(2-2)^3 t = 0\,. \end{cases}$$

A rarefaction wave must therefore be introduced. **[1 point]**

With

$f'(u) = 2(u-2)^3 = v \implies g(v) := (f')^{-1}(v) = \left(\frac{v}{2}\right)^{\frac{1}{3}} + 2$

we have the solution

$$u(x,t) = \begin{cases} 1 & x \le -2t\,, \\ g(\frac{x}{t}) = \left(\frac{x}{2t}\right)^{\frac{1}{3}} + 2 & -2t \le x \le 0 \\ 2 & x \ge 0. \end{cases} \quad \textbf{[2 points]}$$

Exercise 3:

We discuss again the simple traffic flow model from Sheet 1 with the notation introduced there:

$u(x,t)$ = density of vehicles (vehicles/length) at point x at time t,

$v(x,t)$ = velocity at point x at time t,

$q(x,t)$ = flow = number of vehicles passing x at time t per time unit.

We improve our model from Sheet 1 by incorporating maximal density and a maximal velocity

u_{max} = maximal density of vehicles (bumper to bumper),

v_{max} = maximal velocity

This can be done, for example, as follows:

$$v(u(x,t)) = v_{max}\left(1 - \frac{u(x,t)}{u_{max}}\right)$$

a) Set up the continuity equation ($u_t + q_x = 0$).

b) Show again that the characteristics are straight lines and determine their slopes.

c) Sketch the characteristics for

$$v_{max} = 1 \quad \text{(Here has been scaled appropriately!)}$$

$$u(x,0) = \begin{cases} u_l = u_{max}/2 & x < 0 \\ u_r = u_{max} & x > 0 \quad \text{(red traffic light/traffic jam etc.)} \end{cases}$$

d) For the Burgers' equation we allowed shock waves only in the case $u_l > u_r$. There must obviously be a different condition here. What could be the reason for that?

Note: This question can not be answered completely only with help of the lecture slides. You can only make a guess here!

Solution hint to Exercise 3:

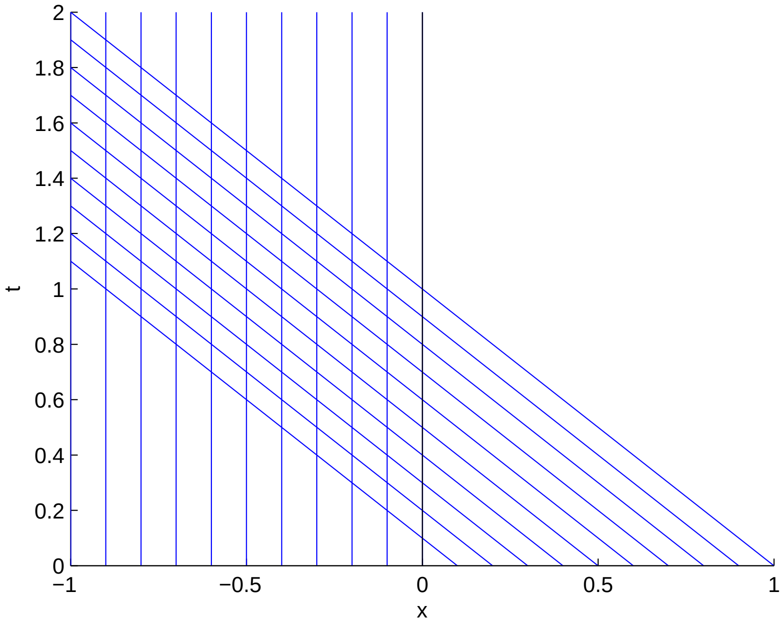
a) $$u_t + \left(v_{max}\, u\,(1 - \frac{u}{u_{max}})\right)_x = u_t + \left(v_{max}\,(u - \frac{u^2}{u_{max}})\right)_x = u_t + (v_{max}\,(1 - \frac{2u}{u_{max}}))u_x = 0$$

b) On the characteristic $x(t)$ the following applies:

$\dot{x}(t) = \left(v_{max}\,(1 - \frac{2u}{u_{max}})\right)$ and $\dot{u}(t) = 0$. The characteristic through a point $(x(0),0)$ has the constant slope in the x–t– plane as a straight line $\left(v_{max}\,(1 - \frac{2u(x(0),0)}{u_{max}}\right)^{-1}$.

The characteristics are straight lines again.

c) Sketch of characteristics:



d) The entropy condition from the lecture is only for convex flow functions f (here q). Since f' is monotonically decreasing, the entropy condition from the lecture does not apply in our case.

What still applies is the graphic interpretation: No information comes out of the shock wave!! So

$$f'(u_l) > \dot{s} > f'(u_r)$$

Since f' is monotonically decreasing, we have the condition for shock waves $u_l < u_r$.