Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Work sheet 2

Exercise 1 (Exam SuSe17, Ex.2a, Hinze/Kiani)

Compute the solution to the following initial value problem for $u(x,t)$:

$$u_t + \frac{1}{2} u_x = -4u, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+,$$
$$u(x,0) = 2\sin(x) \qquad x \in \mathbb{R}\,.$$

Solution: With the method of characteristics one computes:

$\frac{dx}{dt} = \frac{1}{2} \Longrightarrow x(t) = \frac{t}{2} + C \Longrightarrow 2x - t = C$ **[1 point]**

$\frac{du}{dt} = -4u \Longrightarrow \frac{du}{u} == -4\,dt \Longrightarrow \ln(|u|) = -4t + d$

$|u| = e^{-4t} \cdot \tilde{d}, \qquad \tilde{d} \in \mathbb{R}^+$

Since $u = 0$ is also a solution, we get

$u(x(t),t) = D \cdot e^{-4t}, \qquad D \in \mathbb{R} \qquad$ or $\qquad D = u \cdot e^{4t}\,.$ **[2 points]**

Applying the implicit function theorem we have the general solution:

$D = f(C) \Longrightarrow u \cdot e^{4t} = f(2x - t) \Longrightarrow u(x,t) = e^{-4t} \cdot f(2x-t)\,.$ **[1 point]**

The initial condition requires:

$u(x,0) = e^0 \cdot f(2x - 0) \stackrel{!}{=} 2\sin(x) \Longrightarrow f(x) = 2\sin(\frac{x}{2})\,.$ **[1 point]**

$u(x,t) = 2\,e^{-4t} \sin\left(x - \frac{t}{2}\right).$ **[1 point]**

Exercise 2:

Determine the solution $u(x,y)$ to the following differential equation

$$xu_x + \frac{y}{2}u_y = u\,,$$

that satisfies the condition $u(1,y) = 1+y^2\,,\qquad y\in\mathbb{R}\,.$

Solution 2:

a) $xu_x + \frac{y}{2}u_y = u\,,\qquad u(1,y) = 1+y^2$

With x as a parameter one computes for $x\neq 0$ for the equation $u_x + \frac{y}{2x}u_y = \frac{u}{x}$

$$\begin{array}{lll}
\dfrac{dy}{dx} = \frac{y}{2x} & \Longrightarrow & \frac{2dy}{y} = \frac{dx}{x} \\
\Longrightarrow 2\ln|y| = \ln|x|+c & \Longrightarrow & e^{2\ln|y|} = e^{\ln|x|+c} \\
\Longrightarrow y^2 = c_1\cdot x & \Longrightarrow & c_1 = \frac{y^2}{x} \\
\frac{du}{dx} = \frac{u}{x} & \Longrightarrow & \frac{du}{u} = \frac{dx}{x} \\
\ln|u| = \ln|x| + d & \Longrightarrow & u = c_2\cdot x \Longrightarrow c_2 = \frac{u}{x} \\
c_2 = f(c_1) & \Longrightarrow & \frac{u}{x} = f(\frac{y^2}{x}) \Longrightarrow u(x,y) = x\cdot f(\frac{y^2}{x})
\end{array}$$

This is the general solution. Now determine f using the initial condition: Insert $u(1,y) = 1+y^2$ into the general solution

$u(x,y) = x\cdot f(\frac{y^2}{x})$

$u(1,y) = 1\cdot f(\frac{y^2}{1}) = f(y^2) \overset{!}{=} 1+y^2$

Also $f(\mu) = 1+\mu$ and thus

$u(x,y) = x\cdot f(\frac{y^2}{x}) = x\cdot(1+\frac{y^2}{x}) = x+y^2$

One can now determine that the solution for all $x\in\mathbb{R}$ satisfies the ODE + initial condition. So the constraint $x\neq 0$ can be omitted.

ALTERNATIVELY:

Auxiliary problem $\quad xU_x + \frac{y}{2}U_y + uU_u = 0$

$$\begin{cases} \dot{x} = x & \Longrightarrow \quad x = c_1e^t \\ \dot{y} = \frac{y}{2} & \Longrightarrow \quad y = c_2e^{\frac{t}{2}} \\ \dot{u} = u & \Longrightarrow \quad u = c_3e^t \end{cases}$$

It holds (with suitable constants)

$$\begin{cases} x &= cy^2 \\ u &= dx \end{cases}$$

$$c = \frac{x}{y^2}\,,\qquad d = \frac{u}{x}\,,\qquad \phi\left(\frac{x}{y^2},\frac{u}{x}\right) = 0$$

Assuming solvability, we have

$$\frac{u}{x} = \psi\left(\frac{x}{y^2}\right) \qquad u = x \cdot \psi\left(\frac{x}{y^2}\right)$$

additionally should hold $\qquad u(1,y) = \psi\left(\frac{1}{y^2}\right) = y^2 + 1$

$$\Longrightarrow \psi(\mu) = \frac{1}{\mu} + 1 \Longleftrightarrow \psi\left(\frac{x}{y^2}\right) = \frac{y^2}{x} + 1 \Longrightarrow \boxed{u = y^2 + x}$$

Exercise 3: (only for people who compute fast)
Given the following initial value problem

$$u_t + 3u \cdot u_x = 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} 0 & \forall x \leq 0 \\ \frac{1}{3} & \forall x > 0 \end{cases}$$

a) Write down the system of characteristic equations.

b) Are the characteristics straight lines?

c) Draw the characteristics through the points $(x_k, 0) := (k, 0)$ for $k \in \{-3,-2,-1,0,1,2,3\}$.
Compute the values of the solution along these characteristics.

d) Using parts a)-c), can you obtain the values of $u(x,t)$ in the points $(-1,2), (1,2)$ and $(3,2)$?

Solution:

a) Extended problem $\quad U_t + (3u)U_x + 0 \cdot U_u = 0$ implies:

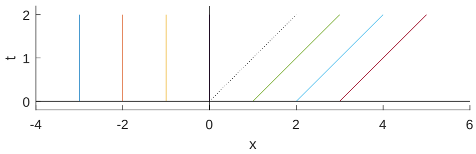
$$\frac{dx}{dt} = 3u, \qquad \frac{du}{dt} = 0 \qquad \Longrightarrow$$

$$u = C, \qquad dx = 3C dt$$

$$\Longrightarrow x(t) = 3Ct + D = 3ut + D \Longrightarrow D = x - 3ut\,.$$

b) The characteristics are straight lines because it holds
$\frac{du}{dt} = 0 \qquad \Longrightarrow u$ is therefore constant along every characteristic. Also, it holds
$\frac{dx}{dt} = 3u \qquad \Longrightarrow$ so $\frac{dx}{dt}$ is constant along the characteristic. i.e.
the slope of the characteristics is constant. These are straight lines.

c) Sketch



d) From the sketch one takes $u(x,t) = 0, \forall x \leq 0$. So in particular, we have $u(-1,2) = 0$.
Furthermore, from the sketch we have $u(x,t) = \frac{1}{3}, \forall x > t$. So we obtain $u(3,2) = \frac{1}{3}$.
The characteristics do not help to determine the solution values $u(x,t)$ for $0 < x < t$, so for example for $u(1,2)$.
The solution to this problem is discussed on the next sheet!

Discussion: 25.04.-29.04.2022