Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 2

Exercise 1: [**5 Points**]

Compute the solution to the following initial value problem for $u(x,t)$:

$$\begin{aligned} u_t - \sin(t)\, u_x &= \cos(t)\,, && x \in \mathbb{R},\, t \in \mathbb{R}^+, \\ u(x,0) &= \exp(-x^2) = e^{-x^2} && x \in \mathbb{R}\,. \end{aligned}$$

Solution: Using method of characteristics one obtains:

$\frac{dx}{dt} = -\sin(t) \implies dx = -\sin(t)dt \implies x = \cos(t) + C_1$ [**1 point**]

$\frac{du}{dt} = \cos(t) \implies du = \cos(t)dt \implies u = \sin(t) + C_2\,.$ [**1 point**]

With $C_1 = x - \cos(t)$ and $C_2 = u - \sin(t)$ we make an Ansatz

$C_2 = f(C_1)$

and obtain

$u - \sin(t) = f(x - \cos(t))$

and hence the general solution is: $u(x,t) = \sin(t) + f(x - \cos(t))\,.$ [**1 point**]

The initial condition requires:

$u(x,0) = \sin(0) + f(x - \cos(0)) = f(x-1) \overset{!}{=} e^{-x^2}\ .$

Also $f(\mu) = e^{-(\mu+1)^2}$ [**1 point**]

$u(x,t) = \sin(t) + e^{-(x-\cos(t)+1)^2}\,.$ [**1 point**]

Exercise 2: **[6= 2+1+2+1 points]**

Given are the following differential equations for $u(x,t)$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$

A) $u_t + 20\, u_x = 21u$.
B) $u_t + 20u\, u_x = 21$.
C) $u_t - 5u^2\, u_x = 0$.
D) $u_t + 5(x+1)\, u_x = 0$.

with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R},$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing and continuously differentiable function.

For which of the differential equations A, B, C, D do the following statements i) and/or ii) hold for the solution of the associated initial value problem?

i) The solution is constant along the characteristics.
ii) The characteristics are straight lines.

Explain your answers. Note that you don't have to compute any solutions!

Solution to exercise 2:

For A) it holds

$\frac{du}{dt} = 21u \Longrightarrow u$ is therefore not constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = 20$.

The slope of the characteristics is therefore constant. So they are straight lines.

For B) it holds

$\frac{du}{dt} = 21 \Longrightarrow u$ is therefore not constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = 20u$.

The slope of the characteristics is therefore not constant. They are not straight lines.

For C) it holds

$\frac{du}{dt} = 0 \Longrightarrow u$ is therefore constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = -5u^2$.

The slope of the characteristics is therefore constant. So they are straight lines.

For D) it holds $\frac{du}{dt} = 0 \Longrightarrow u$ is therefore constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = 5(x+1)$.

The slope of the characteristics is therefore not constant. They are not straight lines.

Exercise 3:

Determine a continuous “solution” $u(x,t)$ to the following initial boundary value problem

$$\begin{aligned} u_t + u_x &= x, \qquad x,t > 0 \\ u(x,0) &= x \qquad (x \geq 0) \\ u(0,t) &= t \qquad (t \geq 0) \end{aligned}$$

using the method of characteristics. To do this, determine the solution to the initial condition $u(x,0) = x$ and to the boundary condition $u(0,t) = t$ and continuously compose these solutions. Is the solution obtained in this way partially differentiable for all $x,t \geq 0$?

Voluntary additional task: If you like, you can do the task too using the Laplace transformation with respect to the variable t. For the transformation x is used as a parameter. In the image space an initial value problem and an ordinary differential equation are to be solved with respect to for x.

Solution 3:

$$\begin{aligned} &\frac{dx}{dt} = 1 \implies x(t) = t + C_1 \implies C_1 = x - t \\ &\frac{du}{dt} = x = C_1 + t \implies u = \frac{t^2}{2} + C_1 t + C_2 \implies C_2 = u - \frac{t^2}{2} - (x-t)t \end{aligned}$$

In case of solvability:

$$\Phi(C_1, C_2) = 0 \Longrightarrow u + \frac{t^2}{2} - xt = f(x-t).$$

For the given initial values we get the solution u_A:

$$u_A(x,0) = u(x,0) = f(x) = x \Longrightarrow u_A(x,t) = (x-t) + xt - \frac{t^2}{2}$$

u_A satisfies the equation and the initial values. However, it holds that $u(0,t) = -t - \frac{t^2}{2}$s. The boundary condition is therefore only fulfilled for $t = 0$. We continue with the general solution

$$u = f(x-t) - \frac{t^2}{2} + xt$$

and plug the boundary values $u(0,t) = t$ in.

$$t = f(-t) - \frac{t^2}{2} \implies f(t) = \frac{t^2}{2} - t \implies u_R(x,t) = \frac{(x-t)^2}{2} - (x-t) - \frac{t^2}{2} + xt$$

If we want to compose the solutions continuously, we have to find a curve along the $u_A = u_R$:

$$\begin{aligned} u_A(x,t) = (x-t) + xt - \frac{t^2}{2} &\stackrel{!}{=} \frac{(x-t)^2}{2} - (x-t) - \frac{t^2}{2} + xt = u_R(x,t) \\ \iff (x-t) &\stackrel{!}{=} \frac{(x-t)^2}{2} - (x-t) \iff (x-t)\left(2 - \frac{(x-t)}{2}\right) \stackrel{!}{=} 0\,. \end{aligned}$$

This condition is fulfilled if and only if $x = t$ or $x = t + 4$. Because of the initial/boundary values, we combine the values along the line $x = t$.

$$u(x,t) := \begin{cases} (x-t) + xt - \dfrac{t^2}{2} & x \geq t \\ \frac{(x-t)^2}{2} - (x-t) + xt - \frac{t^2}{2} & x \leq t\,. \end{cases}$$

As one can easily calculate, the partial derivatives make jumps here. The composite function is therefore not partially differentiable.

Laplace-Transformation

$u(x,t) \circ\!\!-\!\!\bullet\, U(x,s), \qquad u_t \circ\!\!-\!\!\bullet\, sU - u(x,0) = sU - x, \qquad u_x \circ\!\!-\!\!\bullet\, U_x, \qquad t \circ\!\!-\!\!\bullet\, 1/s^2$

New system: $sU - x + U_x = \dfrac{x}{s}, \quad U(0,s) = \frac{1}{s^2}$ gives us a solution to the homogeneous problem $U_h = ce^{-sx}$. Using the variation of constants or special Ansatz we obtain

$$U(x,s) = x\left(\frac{1}{s} + \frac{1}{s^2}\right) - \left(\frac{1}{s^2} + \frac{1}{s^3}\right) + Ce^{-sx}$$

Substituting the boundary value $U(0,s) = 1/s^2$ gives

$$U(x,s) = x\left(\frac{1}{s} + \frac{1}{s^2}\right) - \left(\frac{1}{s^2} + \frac{1}{s^3}\right) + \left(\frac{2}{s^2} + \frac{1}{s^3}\right) e^{-sx}$$

$$U(x,s) \bullet\!\!-\!\!\circ\; x(1+t) - \left(t + \frac{t^2}{2}\right) + h_x(t)\left[2(t-x) + \frac{(t-x)^2}{2}\right] = u(x,t)$$

Submission deadline: 25.04.-29.04.2022