**Fachbereich Mathematik der Universität Hamburg** **SuSe 2022**
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# Differential Equations II for Engineering Students

## Work sheet 1

**Exercise 1:** (Repetition of DGL I)

a) Let $\lambda$ be any fixed real number. Determine a real representation of the general solution to the differential equation
$$y''(t) - \lambda\, y(t) \;=\; 0\,.$$

b) Let $L$ be another fixed positive real number. Determine all solutions to the boundary value problem
$$y''(t) - \lambda\, y(t) \;=\; 0 \quad y(0) \;=\; y(L) \;= 0\,.$$
For which $\lambda \in \mathbb{R}$ does the boundary value problem have nontrivial solutions?

The $\lambda$-values for which there exist non-trivial solutions (i.e. solutions that are not constantly equal to zero) are called eigenvalues of the problem. The corresponding solutions are called eigenfunctions.

**Remark:** *The solutions to this eigenvalue problem will be needed again and again during the semester!*

**Solution hints for the exercise 1:**

a) Following DGL I, we calculate the characteristic polynomial : $\mu^2 - \lambda \;=\; 0$ with the zeros
$$\mu_{1,2} \;=\; \pm\sqrt{\lambda} \quad \Longrightarrow y(t) = \begin{cases} c_1\, e^{\sqrt{\lambda}t} \;+\; c_2\, e^{-\sqrt{\lambda}t} & \lambda \;>\; 0\,, \\ c_1 \;+\; c_2 t & \lambda = 0\,, \\ c_1\, \cos(\sqrt{-\lambda}t) \;+\; c_2\, \sin(\sqrt{-\lambda}t) & \lambda \;<\; 0\,. \end{cases}$$

b) For case $\lambda \;>\; 0$, from the boundary value for $t \;=\; 0$ it follows immediately that $c_2 = -c_1$. The boundary value at $L$ yields:

$c_1\, \left(e^{\sqrt{\lambda}L} \;-\; e^{-\sqrt{\lambda}L}\right) \;=\; 0 \Longrightarrow c_1\, \left(e^{2\sqrt{\lambda}L} \;-\; 1\right) \;=\; 0 \Longrightarrow c_1 = 0$

In this case there exist only the trivial solution $y(t) = 0$

For case $\lambda \;=\; 0$, the solution is a linear function. The only linear function that exists in $t = 0$ and also disappears in $t = L > 0$ is again the trivial solution.

For case $\lambda \;<\; 0$, from the boundary value for $t \;=\; 0$ it follows immediately that $c_1 = 0$. The boundary value in $L$ yields:

$c_2\, \sin(\sqrt{-\lambda}L) = 0 \Longrightarrow c_2 = 0 \,\vee\, \sqrt{-\lambda}L = k\pi$

Non-trivial solutions only exist for $\lambda \;=\; -\left(\frac{k\pi}{L}\right)^2$.

**Exercise 2:** (Repetition of Analysis II)

Determine the appropriate real Fourier series for the following functions:

a) Odd $2L-$ periodic continuation of
$f : [0, 1[\to \mathbb{R}, \quad f(x) = \sin(4\pi x) + 2\sin(6\pi x) \quad L = 1\,.$

b) Even $2L-$ periodic continuation of
$f : [-\frac{\pi}{4}, \frac{5\pi}{4}[\to \mathbb{R}, \quad L = \pi$ with

$$f(t) = \begin{cases} 2, & -\frac{\pi}{4} \le t < \frac{\pi}{4}, \\ 0, & \frac{\pi}{4} \le t < \frac{3\pi}{4}, \\ 2, & \frac{3\pi}{4} \le t < \frac{5\pi}{4}. \end{cases}$$

**Remark:** *For DGL II you will need to know how to calculate Fourier series. Please repeat if necessary!*

**Solution hint to Exercise 2:**

a) Since the function $f(x)$ is continued oddly, a Fourier sine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic sine functions. So we define a series in the form

$$F(x) = \sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{2L}\,x\right)$$

$$L = 1 \implies F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Due to orthogonality relations between the $\sin(k\pi x)$ and $\sin(l\pi x)$ (see Mathe II) and by assuming that the Fourier series is as good as possible approximation of $f$, we have

$$b_4 = 1\,, \quad b_6 = 2\,, \quad b_k = 0 \quad \text{otherwise.}$$

b) Since function $f(t)$ is continued evenly, a Fourier cosine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic cosine functions. In general we define the series as

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{2L}\,t\right).$$

In our special case, we have that the continuation is even $\pi-$ periodic. So we can write the series in the following form

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{\pi}\,t\right).$$

Following Analysis II, and since $T = \pi$, we have for the coefficients

$$\begin{aligned} a_k &= \frac{4}{\pi} \int_0^{\frac{T}{2}} f(t) \cos(k\omega t)\, dt \\ &= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} 2 \cdot \cos\left(k \frac{2\pi}{\pi} t\right) dt + \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 0 \cdot \cos\left(k \frac{2\pi}{\pi} t\right) dt \\ &= \frac{8}{\pi} \int_0^{\frac{\pi}{4}} \cos(2kt) dt \end{aligned}$$

For $k = 0$ it holds

$$a_0 = \frac{8}{\pi} \int_0^{\frac{\pi}{4}} 1 dt = \frac{8}{\pi} [t]_0^{\frac{\pi}{4}} = 2$$

and for $k > 0$ we obtain

$$a_k = \frac{8}{\pi} \int_0^{\frac{\pi}{4}} \cos(2kt) dt = \frac{4}{\pi} \left[ \frac{1}{k} \sin(2kt) \right]_0^{\frac{\pi}{4}} = \frac{4}{\pi k} \sin\left(\frac{k\pi}{2}\right).$$

Hence

$$a_k = \begin{cases} 2 & k = 0 \\ 0 & k = 2m,\ m \in \mathbb{N} \\ \frac{4(-1)^m}{\pi(2m+1)} & k = 2m+1,\ m \in \mathbb{N}_0 \end{cases}$$

so

$$a_0 = 2 \qquad a_1 = \frac{4}{\pi} \qquad a_3 = -\frac{4}{3\pi} \qquad a_5 = \frac{4}{5\pi} \quad \cdots$$

The first four non-vanishing summands of the Fourier series are e.g.

$$1 + \frac{4}{\pi} \cos(2t) - \frac{4}{3\pi} \cos(6t) + \frac{4}{5\pi} \cos(10t).$$

**Discussion: 11-15.04.2022**