Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1: (Repetition Analysis II)

For the derivation of parameter-dependent integrals for sufficiently smooth f holds the **Leibniz–Rule** :

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))}$$

Find the derivative of the function $F(x)$ defined as

$$F(x) := \int_{-x}^{x^2} e^{xt}\,dt$$

and compute $\lim\limits_{x\to 0} F'(x)$.

Solution to Exercise 1:

$$F(x) = \int_{-x}^{x^2} e^{xt}\,dt\,, \qquad b(x) := x^2,\ a(x) := -x,\ f(t,x) := e^{xt}$$

$$b'(x) = 2x \qquad a'(x) = -1$$

$$f(b(x),x) = e^{x^3} \qquad f(a(x),x) = e^{-x^2}$$

$$\begin{aligned}
F'(x) &= \int\limits_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)\,dt + b'(x)f(b(x),x) - a'(x)f(a(x),x) \\
&= \int\limits_{-x}^{x^2} te^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\
&= \left[\frac{t}{x}e^{tx}\right]_{-x}^{x^2} - \int\limits_{-x}^{x^2} \frac{1}{x}e^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\
&= 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left[e^{tx}\right]_{-x}^{x^2} = 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left(e^{x^3} - e^{-x^2}\right)
\end{aligned}$$

Substitution/ L'Hospital gives:

$$F'(0) = 0 + 2 - \lim_{x\to 0} \frac{1}{x^2}(e^{x^3} - e^{-x^2}) = 2 - \lim_{x\to 0} \frac{3x^2e^{x^3} + 2xe^{-x^2}}{2x} = 2 - 1 = 1\,.$$

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$$\begin{aligned} u(x,t) &= \text{(length-)density of the vehicles at the point } x \text{ at the time } t \\ &= \text{vehicles/unit length at point } x \text{ at the time } t \\ v(x,t) &= \text{ speed at the point } x \text{ at the time } t, \\ q(x,t) &= u(x,t)\cdot v(x,t) = \text{ flow} \\ &= \text{ amount of vehicles passing the point } x \text{ at the time } t \text{ per unit time} \end{aligned}$$

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a space interval $[a, a+\Delta a]$ at the time t.

Then on the one hand it holds that

$$N(t,a,\Delta a) = \int_a^{a+\Delta a} u(x,t)\,dx$$

and on the other hand it also holds

$$N(t,a,\Delta a) - N(t_0,a,\Delta a) = \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$$

Derive from this the so-called conservation equation for the mass (number of vehicles)

$$u_t + q_x = 0.$$

Hints on how to proceed:

- Derive both formulas for N with respect to t. Please note that for the derivation of parameter-dependent integrals with sufficiently smooth f holds the **Leibniz rule**:

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))}$$

- Divide by Δa.
- Consider the limit $\Delta a \to 0$.

b) Additionally assume that the velocity depends only on the density: $v = v(u)$. Show that in this case the equation

$$\frac{\partial u}{\partial t} + \frac{dq}{du}\cdot\frac{\partial u}{\partial x} = 0$$

describes the conservation of mass.

c) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive.

$$v(x,t) = c + \frac{k}{u(x,t)}$$

What is the continuity equation (=conservation equation for the mass)?

d) Solve the continuity equation derived in part c) for $c = 3$ and the initial condition $u(x,0) = e^{-x^2}$.

Show that every sufficiently smooth function $u(x,t) = f(x-ct)$ solves the differential equation. Define f such that the initial condition is satisfied.

Solution:

a) On the one hand, it holds $\quad N(t) = \int_a^{a+\Delta a} u(x,t)\,dx$

and on the other hand $\quad N(t) - N(t_0) = \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau) d\tau\,.$

Differentiating with respect to t gives

$$\frac{\partial}{\partial t} N(t) = \frac{\partial}{\partial t} \int_a^{a+\Delta a} u(x,t)\,dx = q(a,t) - q(a+\Delta a,t)$$

Letting Δa to zero, and with sufficient smoothness of the functions, we have

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_a^{a+\Delta a} \frac{\partial}{\partial t} u(x,t)\,dx = \lim_{\Delta a \to 0} -\frac{q(a+\Delta a,t) - q(a,t)}{\Delta a}$$

$$\Longrightarrow \frac{\partial}{\partial t} u(a,t) = -\frac{\partial}{\partial a} q(a,t).$$

Since these considerations hold at every point, we have the continuity equation

$u_t + q_x = 0\,.$

b) Actually is straightforward, since in this case we have $q(x,t) = u(x,t) \cdot v(u(x,t))$. The flow q is therefore a function of $u(x,t)$. The assertion then follows from the chain rule.

In more details:

With $q(x,t) = u(x,t) \cdot v(u(x,t))$ we have

$$\frac{dq}{du} \cdot \frac{\partial u}{\partial x} = \frac{d}{du}(u \cdot v(u)) \cdot u_x = (v(u) + u \cdot v_u) \cdot u_x$$

and on the other hand it holds

$$\frac{\partial}{\partial x} q(x,t) = \frac{\partial}{\partial x}(u(x,t) \cdot v(u(x,t))) = u_x \cdot v(u) + u \cdot v_u \cdot u_x\,.$$

c)

$$v(x,t) \; = \; c \; + \; \frac{k}{u(x,t)} \qquad q(x,t) \; = \; c \cdot u(x,t) \; + \; k$$

From the continuity equation from part b) we have

$$\frac{\partial u}{\partial t} + \frac{dq}{du} \cdot \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0$$

The linear transport equation is thus obtained.

d) We have to solve the equation from part c)

$u_t + 3u_x + = 0.$

With the ansatz $u(x,t) = f(x - 3t)$ it holds

$u_t(x,t) = f'(x - 3t) \cdot (-3), \qquad u_x(x,t) = f'(x - 3t)$

and hence $u_t + 3u_x \; = \; 0\,.$

The initial condition requires:

$u(x,0) = f(x) \overset{!}{=} e^{-x^2} \implies u(x,t) \; = \; f(x - 3t) \; = \; e^{-(x-3t)^2}.$

Note : This is a very simple, linearized model. For example, it allows for any density and any speed. A somewhat more realistic problem would already produce shock and rarefaction waves (see later exercises).

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