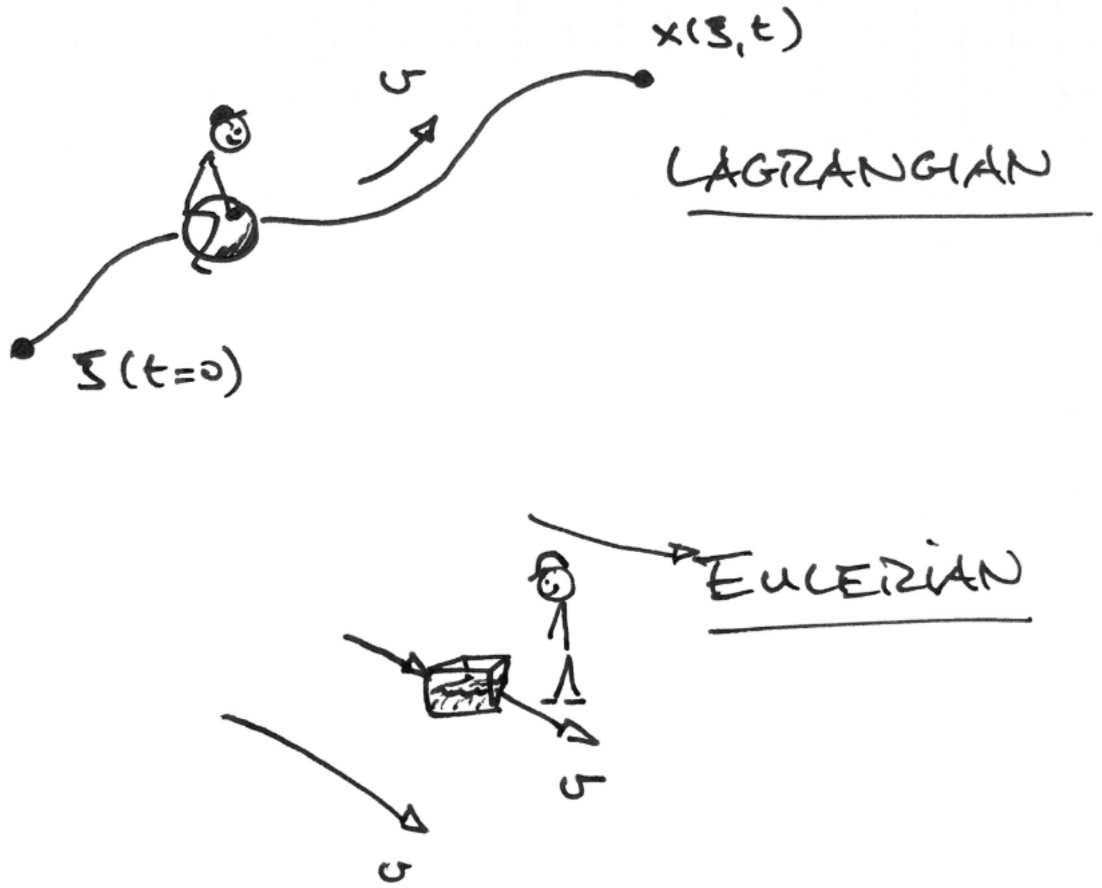
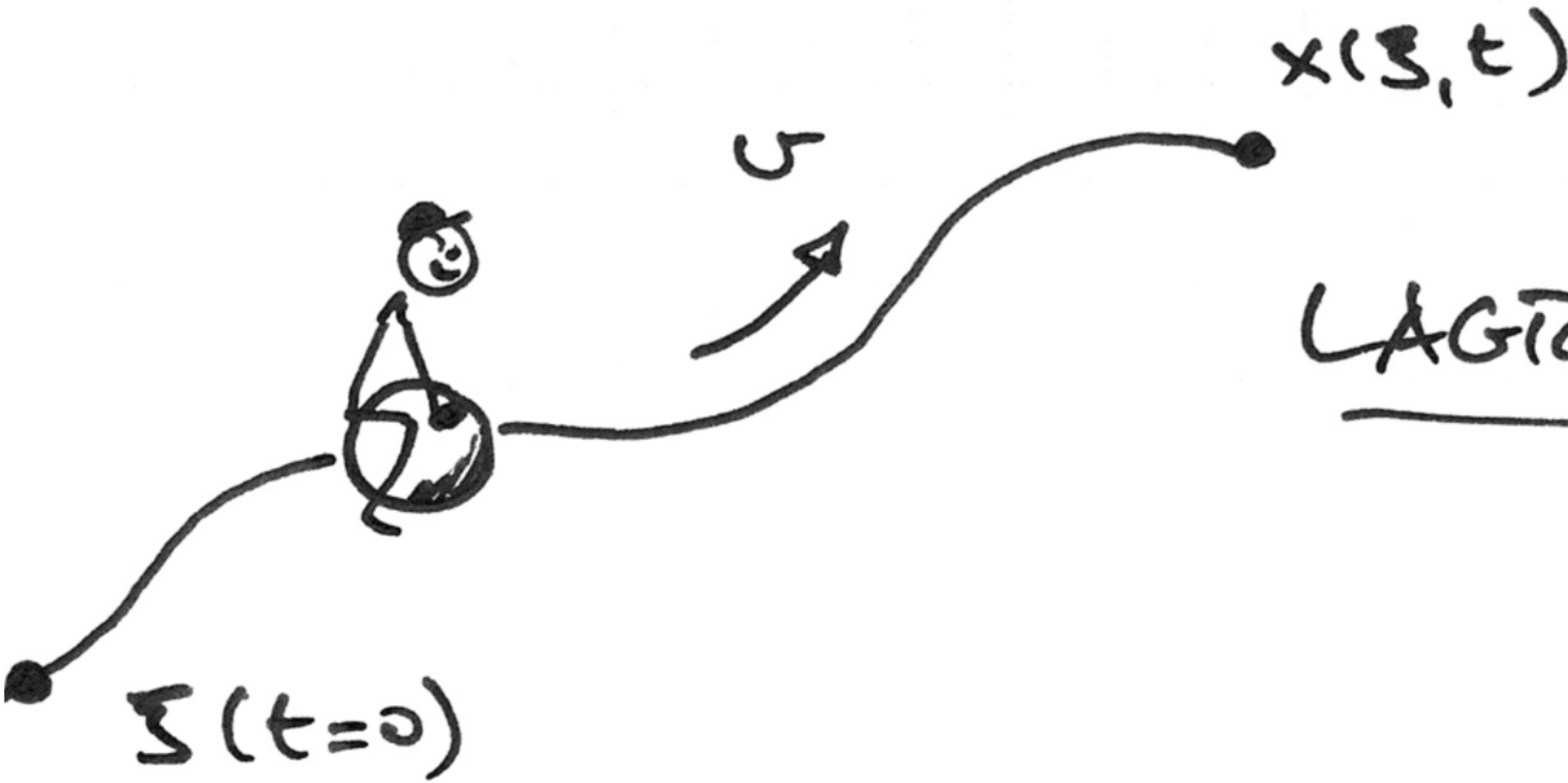
Differentialgleichungen II

Sommer 2018

Numerische Lösung der Transportgleichung:
Lagrange und Semi-Lagrange Verfahren

Lagrangesche Perspektive





LAGRANG

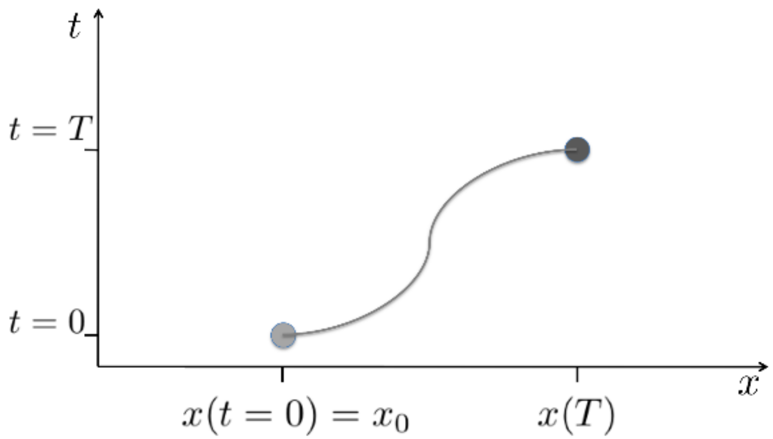
Formalization

- Position: $x = x(t)$.
- Velocity: $v = v(x, t)$.

Particle position can be computed by

$$\dot{x} = \frac{dx}{dt} = v(x, t)$$

With initial condition $x(t = 0) = x_0$

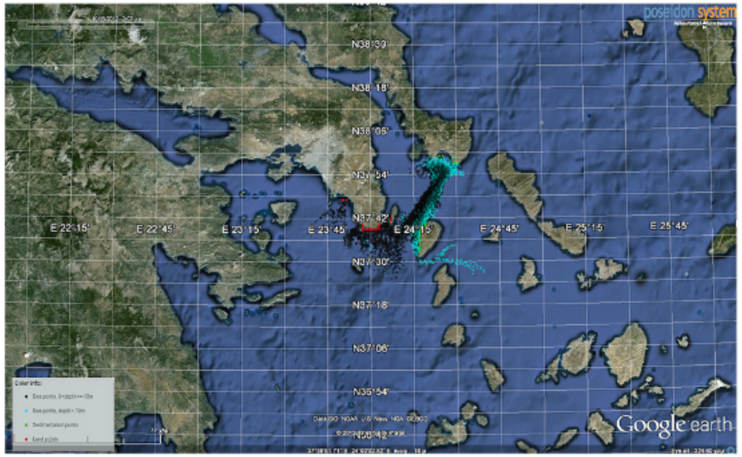


Lagrangian Transport with Source

The equation is given by

$$\frac{d\rho}{dt} = s(x,t)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot \frac{dx}{dt}$



http://www.medess4ms.eu/oil-spill-models

Remark: With $\dot{x} = v(x,t)$ we have that

$$\frac{d}{dt} = \rho_t + v\rho_x = s(x,t).$$

Summary: We need to solve two ODEs:

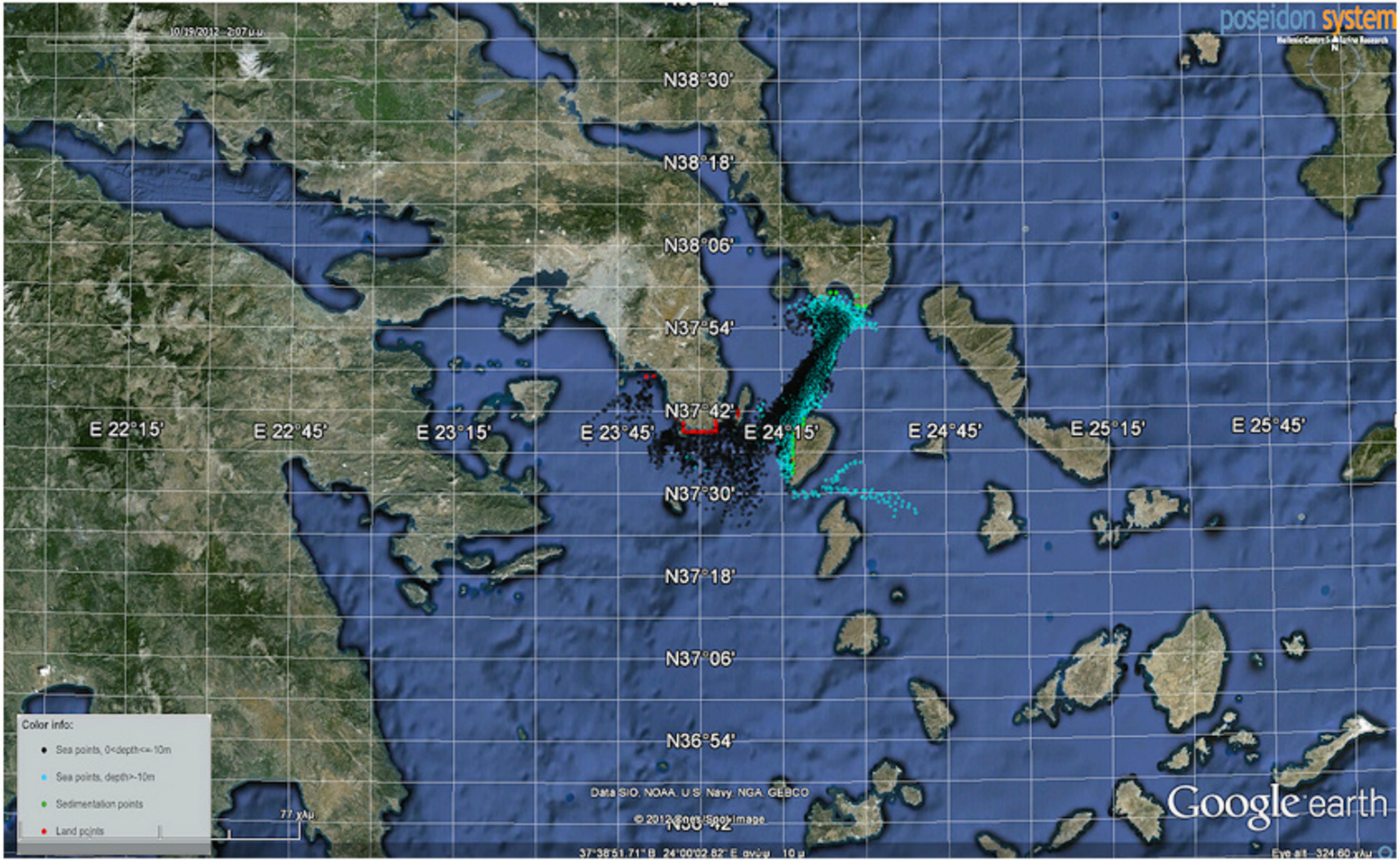
$$\begin{aligned} \dot{x} &= v(x,t), \quad x(0) = x_0, \\ \dot{\rho} &= s(\rho,x,t), \quad \rho(x,0) = \rho_0(x). \end{aligned}$$

Remark: Usually many particles are used for real applications!

Remark: Homogeneous advection, where $s \equiv 0$, yields:

$$\dot{\rho} = 0 \quad \Rightarrow \quad \rho(\cdot,t) \equiv \text{const.} = \rho_0(\cdot).$$

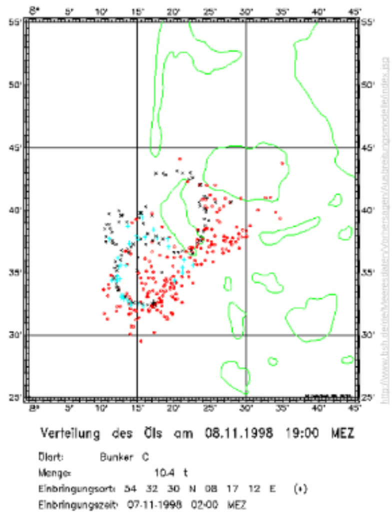
Since $x = x(t)$, the particle position is implied: $\rho(x,t) = \rho(x(t),t))$.

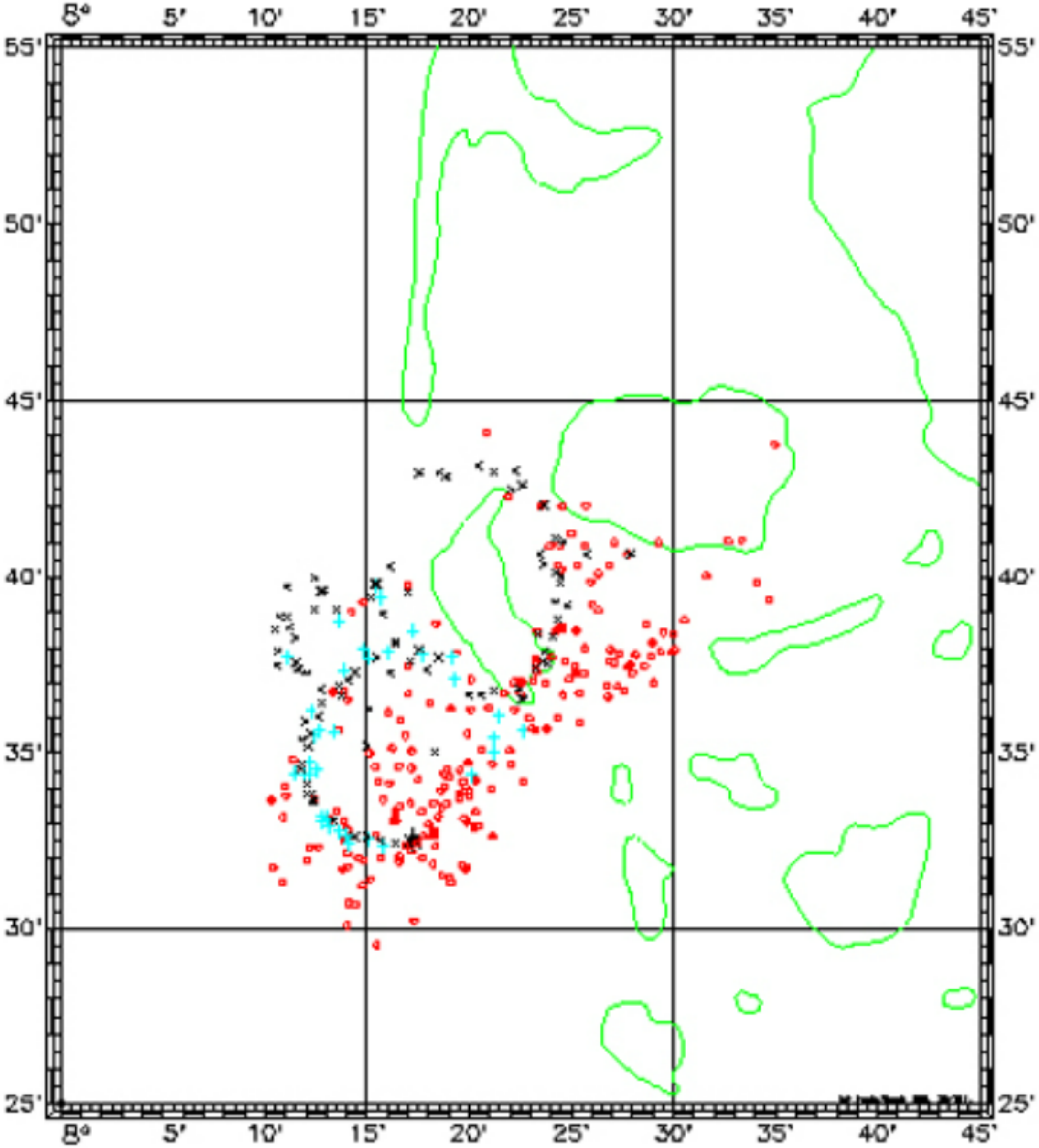


http://www.medess4ms.eu/oil-spill-models

Problems of purely Lagrangian Methods

- Distribution of particles will eventually become very irregular.
- Interaction between particles is difficult to simulate (diffusive processes).
- Density distribution fields with spatial coverage hard to reconstruct.





Verteilung des Öls am 08.11.1998 19:00 MEZ

Ölart: Bunker C

Menge: 10.4 t

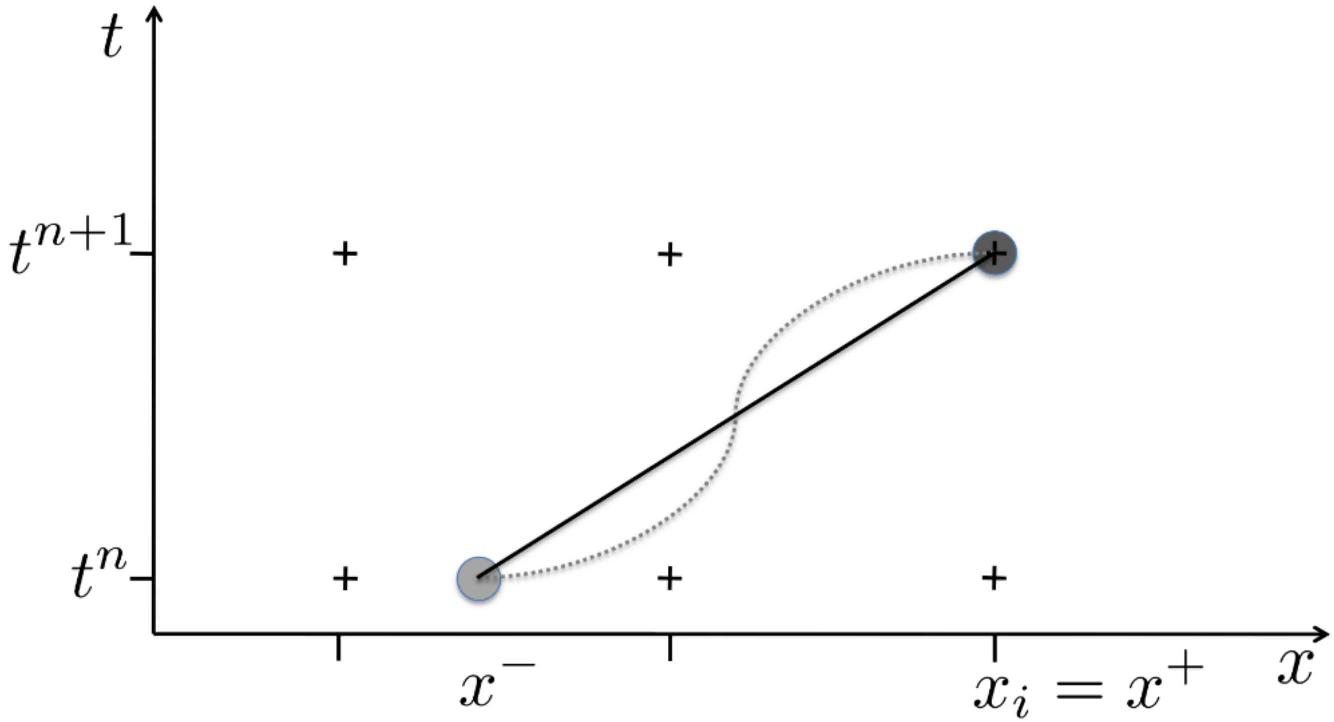
Einbringungsort: 54 32 30 N 08 17 12 E (•)

Einbringungszeit: 07-11-1998 02:00 MEZ

Idea

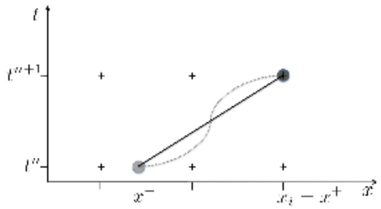
Combine Lagrangian and Eulerian Methods

Idee der Semi-Lagrange Methode



Formalisierung

Problem: Passive advection ($s \equiv 0$):



$$\frac{dx}{dt} = v(x,t), \quad x(0) = x_0,$$
$$\frac{d\rho}{dt} = 0, \quad \rho(x,0) = \rho_0(x).$$

Strategy:

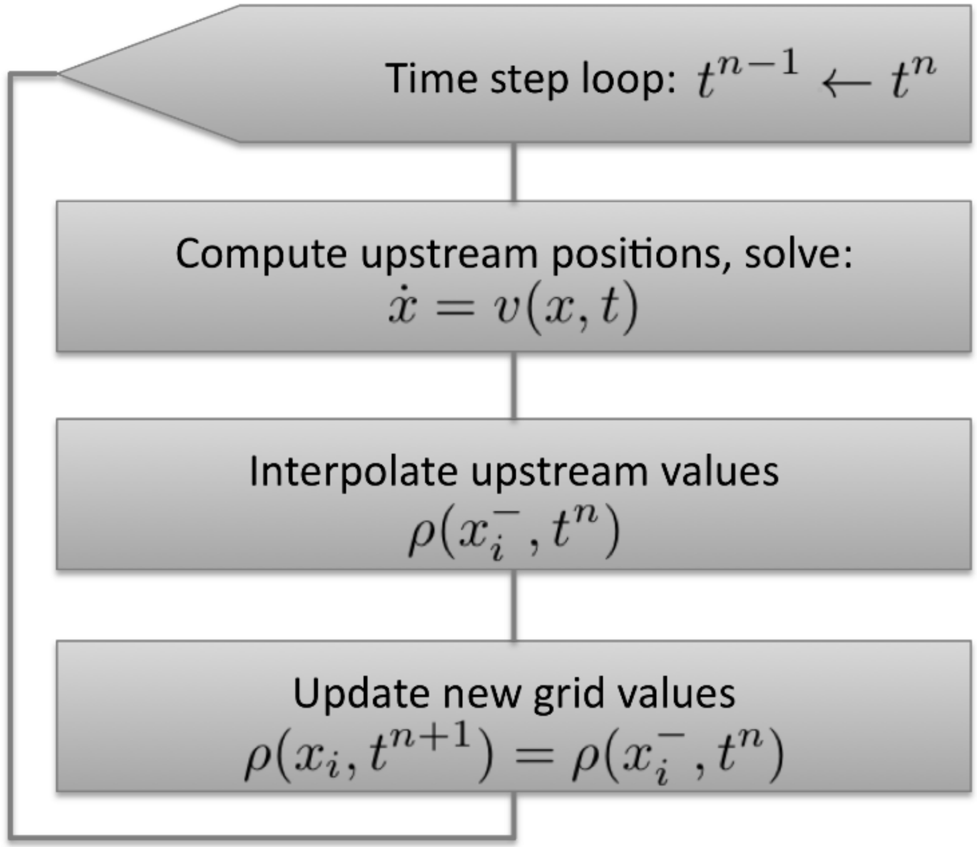
- Solve $\frac{dx}{dt} = v$ by any ODE solver,
- Solve $\frac{d\rho}{dt} = 0$ by finite difference.

$$\frac{d\rho}{dt} \approx \frac{\rho(x_i, t^{n+1} - \rho(x_i^-, t^n)}{\Delta t} = 0$$

$$\Rightarrow \rho^+ = \rho^-.$$

x_i, $i = 1 : N$ grid points, t^n, $n = 1 : M$ time steps.

Algorithmus



Stabilität und Konsistenz

Von Neumann Stability Analysis:

Assume linear interpolation of upstream points and exact wind, i.e.

$$\rho_i^{n+1} = (1-\nu)\rho_{k_i}^n + \nu\rho_{k_i-1}^n$$

- $[x_{k_{i-1}}, x_{k_i}]$ interval containting upstream point x_i^-,
- $\nu = \frac{x_k - x^-}{\Delta x}$.

Then using $z_n e^{ik(jh)}$ we have:

$$\begin{aligned} z_{n+1}e^{ik(jh)} &= (1-\nu)z_n e^{ik(k_i h)} + \nu z_n e^{ik(k_i - 1h)}, \\ &= z_n \left[1 - \nu(1 - e^{-ik(h)})\right] e^{ik(k_i - j)h} e^{ik(jh)}; \\ \Rightarrow \xi &= \quad \left[1 - \nu(1 - e^{-ik(h)})\right] e^{ik(k_i - j)h}; \\ \Rightarrow |\xi|^2 &= 1 - 2\nu(1-\nu)(1 - \cos(kh)). \end{aligned}$$

Stability follows for: $0 \leq \nu \leq 1$, i.e. always!

Remark (Order of Consistency):
It can be shown the the semi-Lagrangian advection scheme retains the consistency order of the discretization schemes involved:

$$\left.\begin{array}{l} \dot{x} = v \text{ order } p \\ \dot{\rho} = 0 \text{ order } p \end{array}\right\} \Rightarrow \text{SLM order } p.$$

Stabilität und Konsisten

Von Neumann Stability Analysis:

Assume linear interpolation of upstream points and exact wind, i.e.

$$\rho_i^{n+1} = (1-\nu)\rho_{k_i}^n + \nu\rho_{k_i-1}^n$$

- $[x_{k_{i-1}}, x_{k_i}]$ interval contain
- $\nu = \frac{x_k - x^-}{\Delta x}$.

Then using $z_n e^{ik(jh)}$ we have:

$$\begin{aligned} z_{n+1}e^{ik(jh)} &= (1-\nu)z_n e^{ik(k_i h)} + \nu z_n e^{ik(k_i-1h)}, \\ &= z_n \left[1-\nu(1-e^{-ik(h)})\right] e^{ik(k_i-j)h} e^{ik(jh)}; \\ \Rightarrow \xi = &\quad \left[1-\nu(1-e^{-ik(h)})\right] e^{ik(k_i-j)h}; \\ \Rightarrow |\xi|^2 &= 1-2\nu(1-\nu)(1-\cos(kh)). \end{aligned}$$

Stability follows for: $0 \leq \nu \leq 1$, i.e. always!

ct wind, i.e.

- $[x_{k_{i-1}}, x_{k_i}]$ interval containting upstream point x_i^-,
- $\nu = \frac{x_k - x^-}{\Delta x}$.

$$\rho_i = (1-\nu)\rho_{k_i} + \nu\rho_{k_i-1}$$

Then using $z_n e^{ik(jh)}$ we have:

$$\begin{aligned}
z_{n+1}e^{ik(jh)} &= (1-\nu)z_n e^{ik(k_i h)} + \nu z_n e^{ik(k_i - 1h)}, \\
&= z_n \left[1 - \nu(1 - e^{-ik(h)})\right] e^{ik(k_i - j)h} e^{ik(jh)}; \\
\Rightarrow \xi &= \quad \left[1 - \nu(1 - e^{-ik(h)})\right] e^{ik(k_i - j)h}; \\
\Rightarrow |\xi|^2 &= 1 - 2\nu(1-\nu)(1 - \cos(kh)).
\end{aligned}$$

$\nu \leq 1$, i.e. always!

Then using $z_n e^{ik(jh)}$ we have:

$$z_{n+1}e^{ik(jh)} = (1-\nu)z_n e^{ik(k_i$$

$$= z_n \Big[1 - \nu(1 - e$$

$$\Rightarrow \xi = \Big[1 - \nu(1 - e$$

$$\Rightarrow |\xi|^2 = 1 - 2\nu(1-\nu)($$

Stability follows for: $0 \le \nu \le 1$, i.e. always!

Remark (Order of Consistency):
It can be shown the the semi-Lagrangi
the consistency order of the discretizat

$$\dot{x} = v \text{ order } p$$
$$\dot{\rho} = 0 \text{ order } p$$

$$\Rightarrow |\xi|^2 = 1 - 2\nu(1-\nu)(1-\cos(kh)).$$

for: $0 \leq \nu \leq 1$, i.e. always!

Remark (Order of Consistency):
It can be shown the the semi-Lagrangian advection scheme retains the consistency order of the discretization schemes involved:

$$\left.\begin{array}{l} \dot{x} = v \text{ order } p \\ \dot{\rho} = 0 \text{ order } p \end{array}\right\} \Rightarrow \text{SLM order } p.$$

Advektion mit Quellterm

Recall:

$\rho_t + v\rho_x = s.$

Lagrangian:

$\dot{x} = v; \dot{\rho} = s.$

For one particle we solve:

$$\begin{aligned} \dot{x} &= v(x,t), \quad x(0) = x_0, \\ \dot{\rho} &= s(\rho, x, t), \quad \rho(x,0) = \rho_0(x). \end{aligned}$$

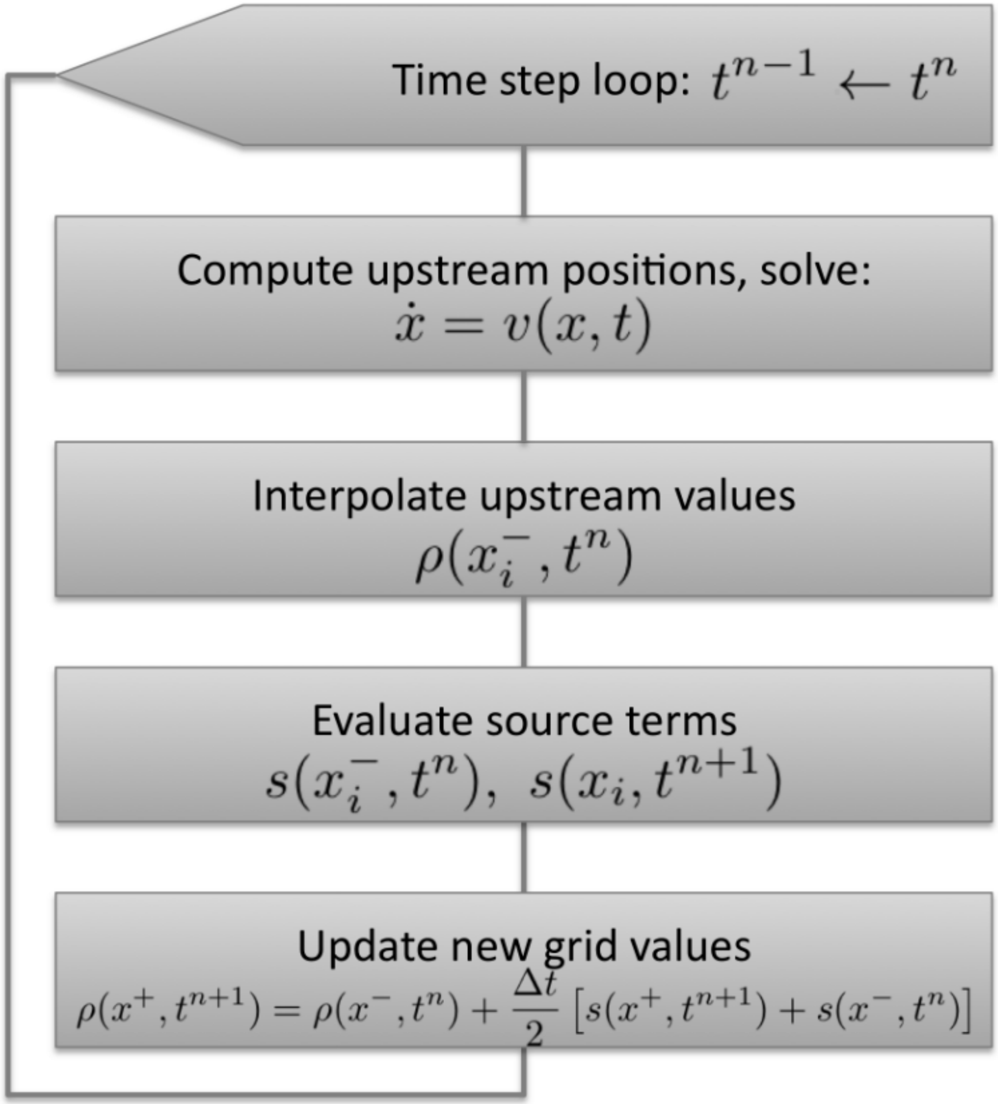
Assume $s = s(x,t)$, use trapezoidal rule:

$$\frac{\rho(x^+, t^{n+1}) - \rho(x^-, t^n)}{\Delta t} = \frac{1}{2}\left[s(x^+, t^{n+1}) + s(x^-, t^n)\right]$$

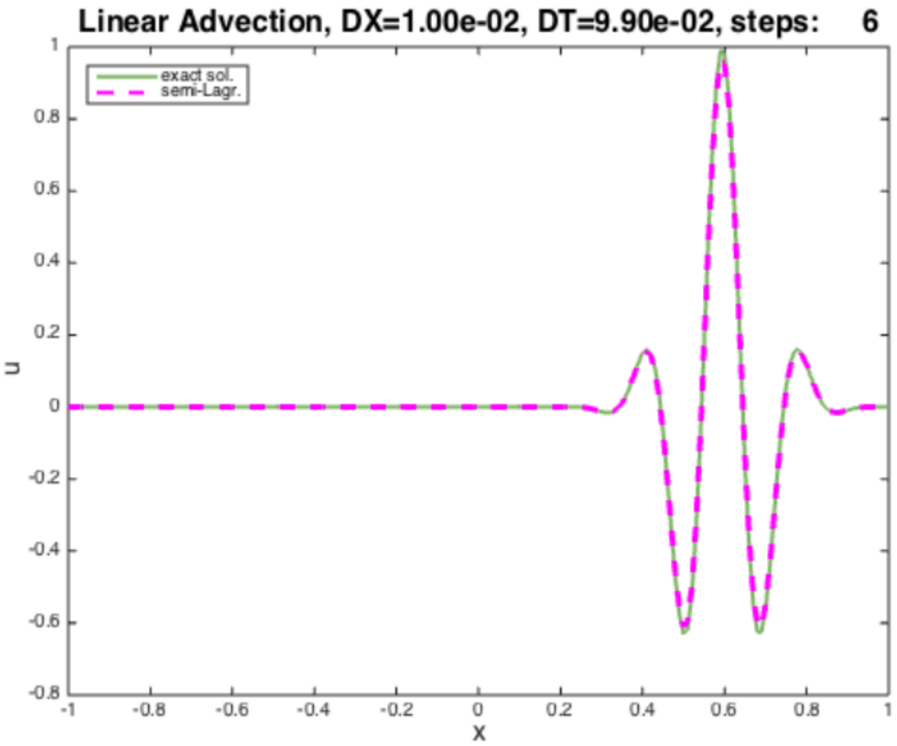
Or even simpler:

$$\frac{\rho(x^+, t^{n+1}) - \rho(x^-, t^n)}{\Delta t} = s(x^0, t^{n+1/2})$$

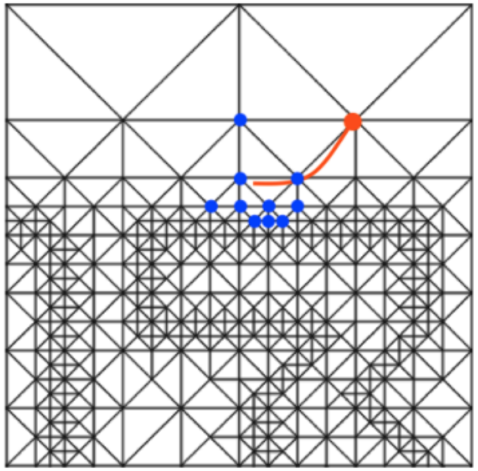
Algorithmus mit Quellterm



Semi-Lagrangian Algorithm Result



Problem bei nicht-uniformen Gittern



Semi-Lagrange-Methode:

Lagrange Form: $\frac{dc}{dt} = 0$

Differenz entlang Trajektorie:

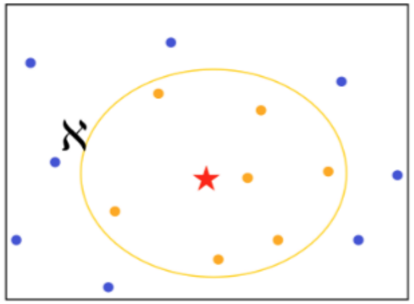
$$\frac{dc}{dt} \approx \frac{c(\vec{x}, t + \Delta t) - c(\vec{x} - 2\vec{\alpha}, t - \Delta t)}{2\Delta t}$$

$$\Rightarrow \boxed{c(\vec{x}, t + \Delta t)} = \boxed{c(\vec{x} - 2\vec{\alpha}, t - \Delta t)}$$

Interpolation mit radialen Basisfunktionen

Interpolationsproblem:

$s : \mathbb{R}^d \to \mathbb{R}$ mit $\quad s|_{\aleph} = c(\cdot, t)|_{\aleph}.$



ℵ Menge der k Nachbarn

Interpolierende Funktion:

$$s(x) = \sum_{j=1}^{k} \lambda_j \Phi(\|x-y_j\|)+p(x), \quad y_j \in \aleph, \; p(x) = \sum_{l=1}^{q} \mu_l p_l(x).$$

$$\begin{bmatrix} A_{\Phi,\aleph} & P \\ P & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \begin{bmatrix} c(y_j) \\ 0 \end{bmatrix}$$

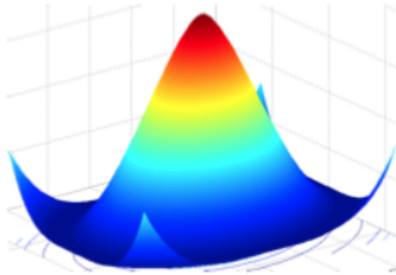
$$A_{\Phi,\aleph} = \left[\Phi(\|x_j - x_l\|)\right]_{j,l=1:k}, \quad P = \left[p_n(x_j)\right]_{n=1:q,j=1:k}.$$

(ℵ nicht-degeneriert)

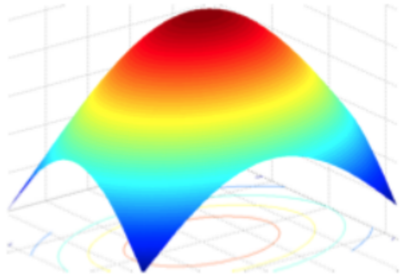
R.L. Hardy (1971), J. Duchon (1976), W.R. Madych/S.A. Nelson (1988)

Beispiele radialer Basisfunktionen

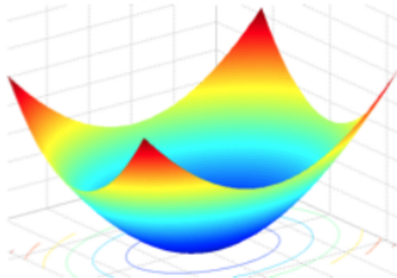
Thin Plate Spline $\Phi(r) = r^2 \log r$



Gaussians $\Phi(r) = e^{-r^2}$

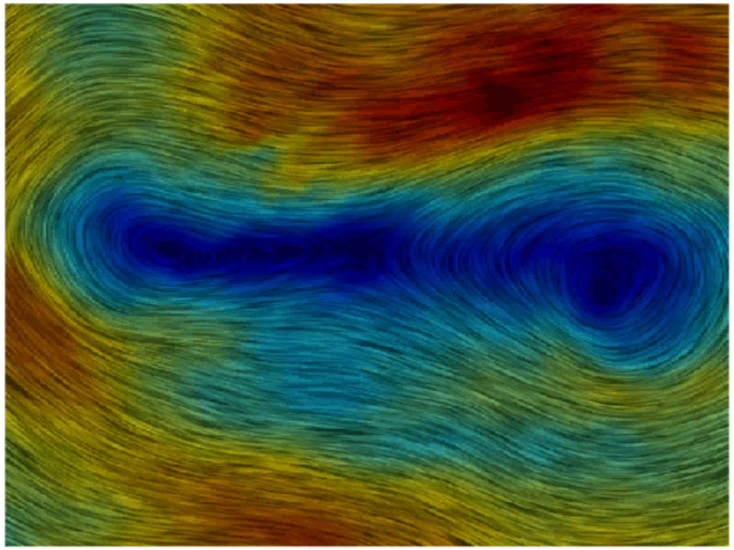


Multiquadrics $\Phi(r) = \sqrt{r^2 + 1}$

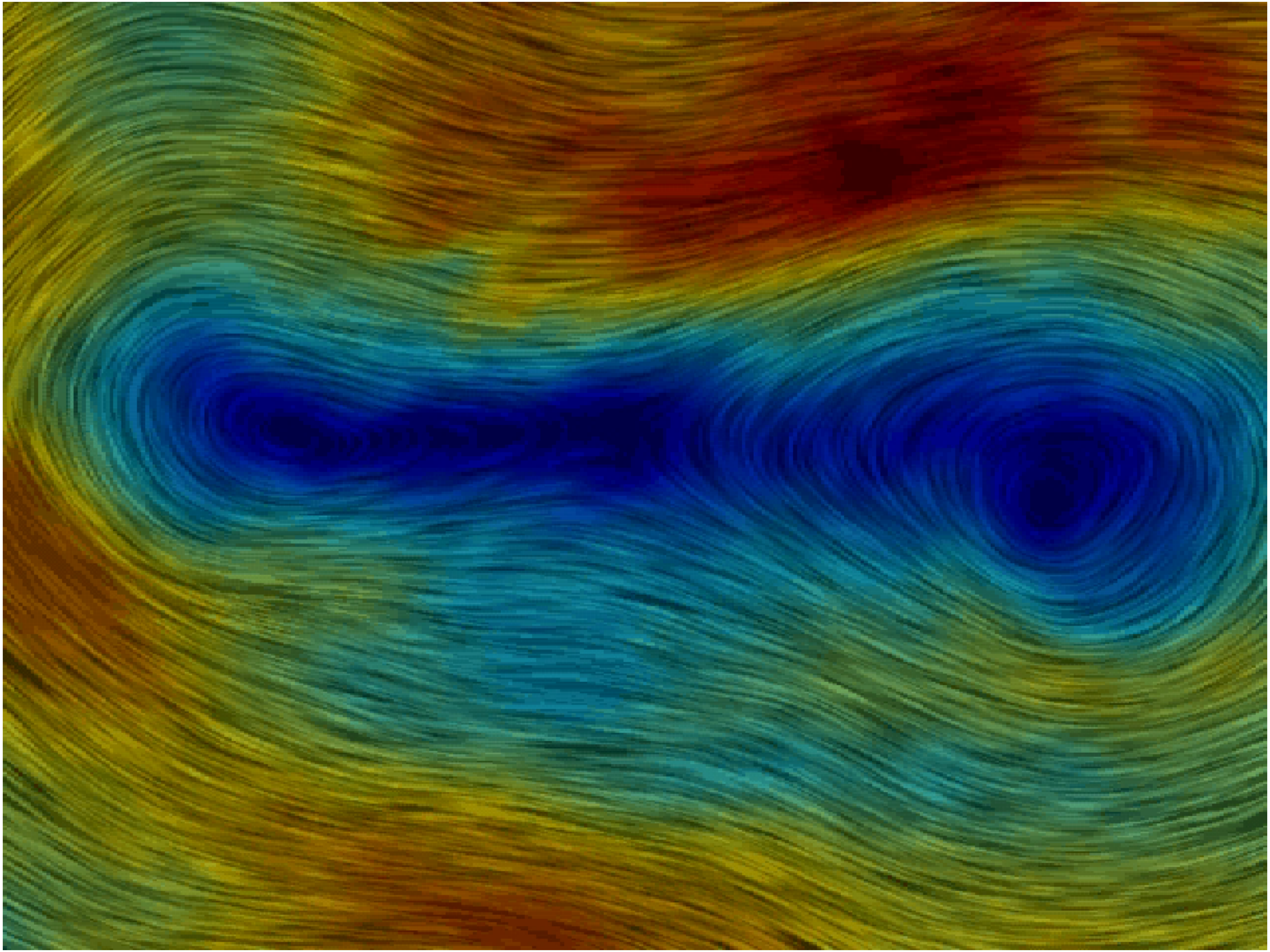


Anwendungsbeispiel: Spurenstofftransport

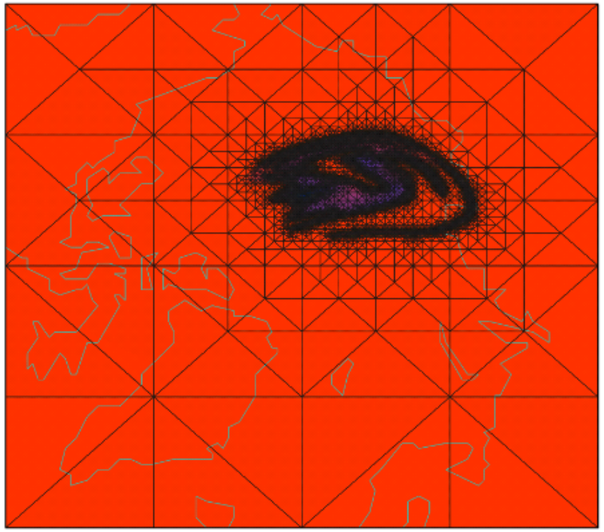
Gegeben: Wind-Daten

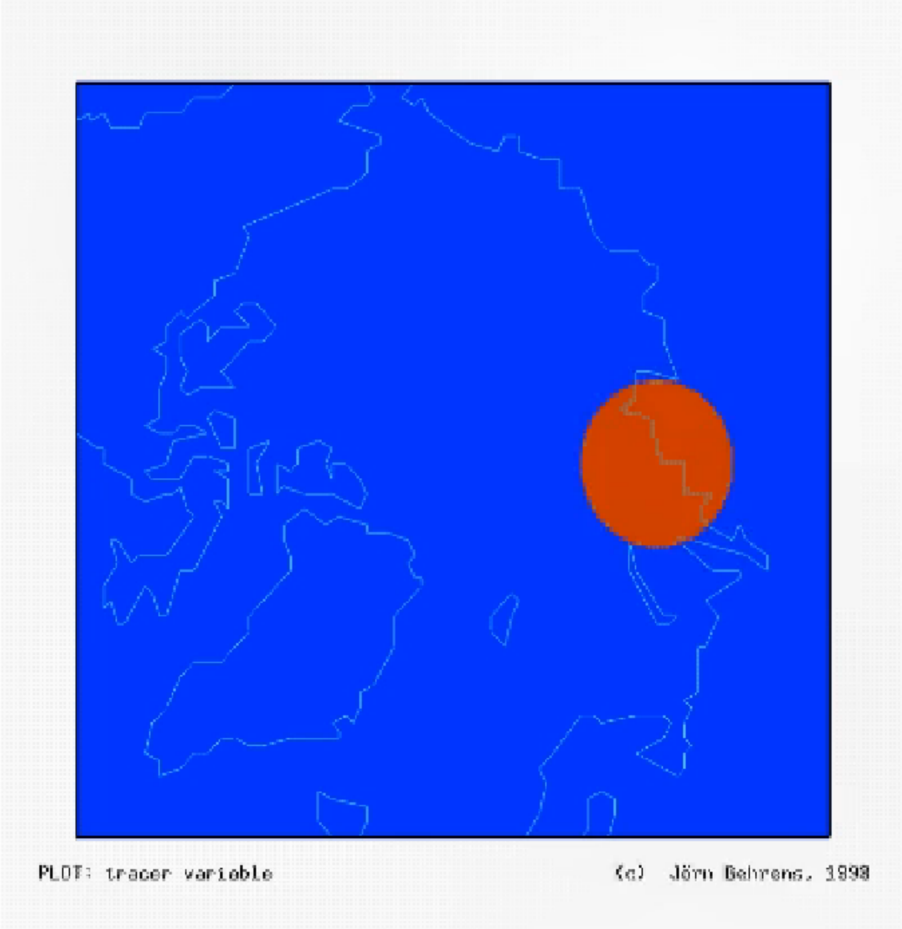


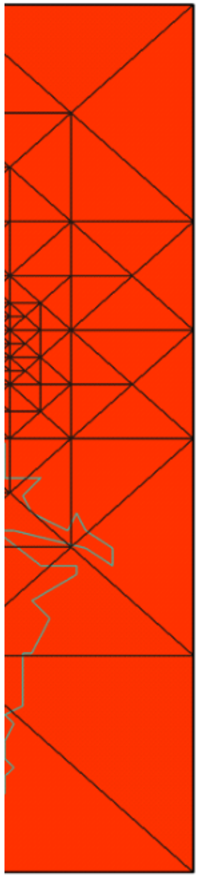
http://www.lib.utexas.edu/maps

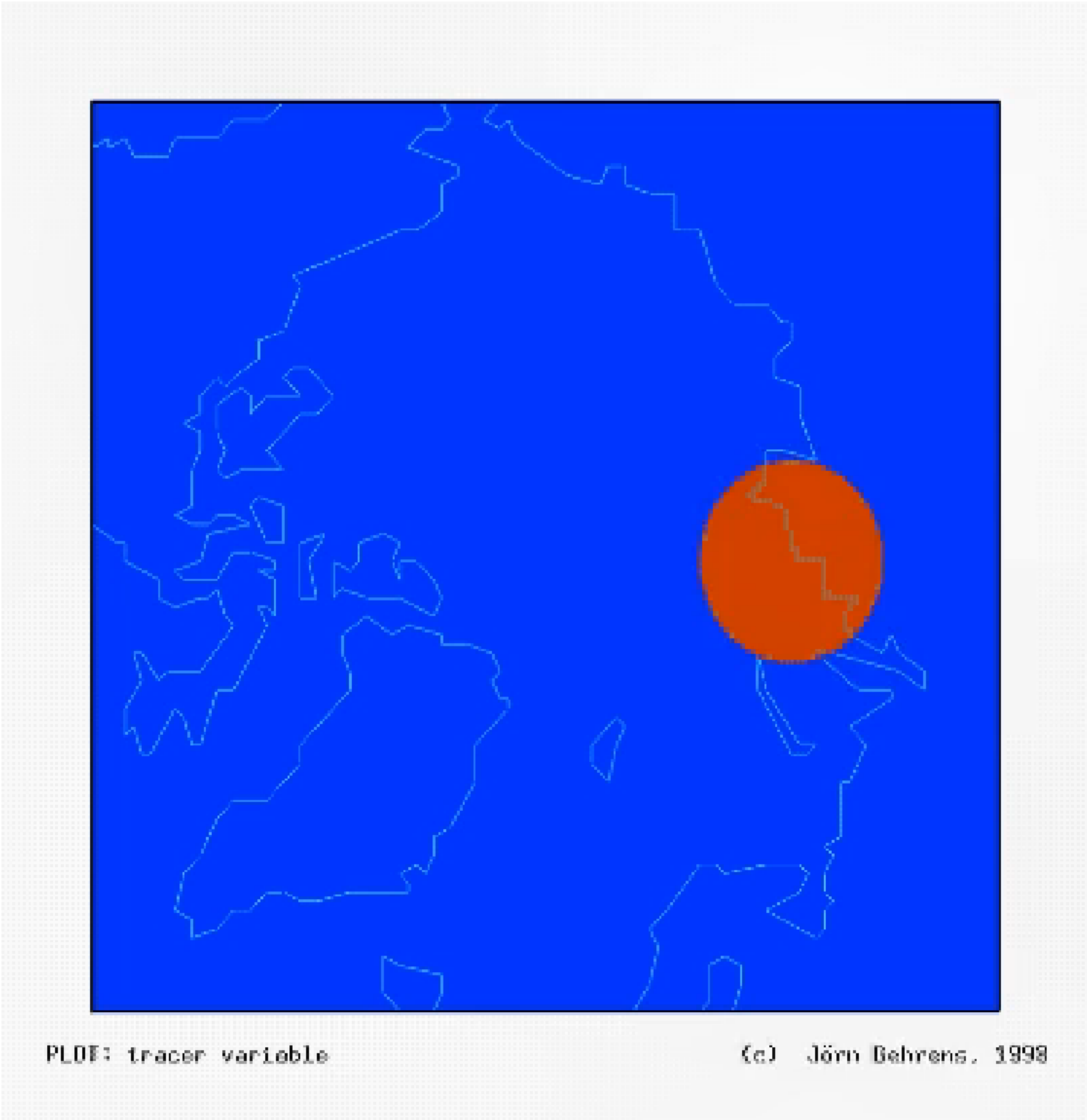


Spurenstofftransport (adaptives Gitter)

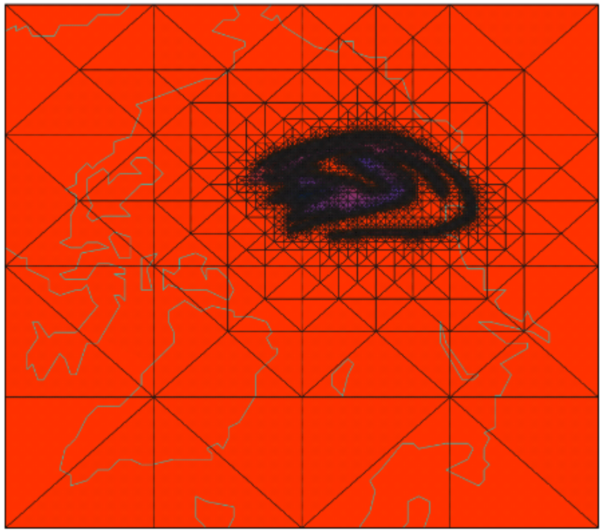


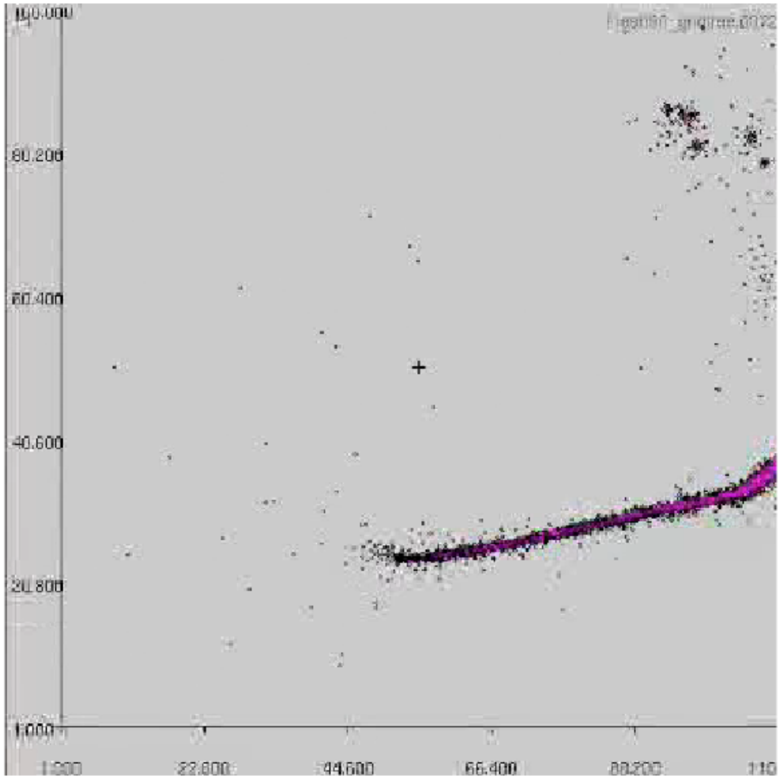


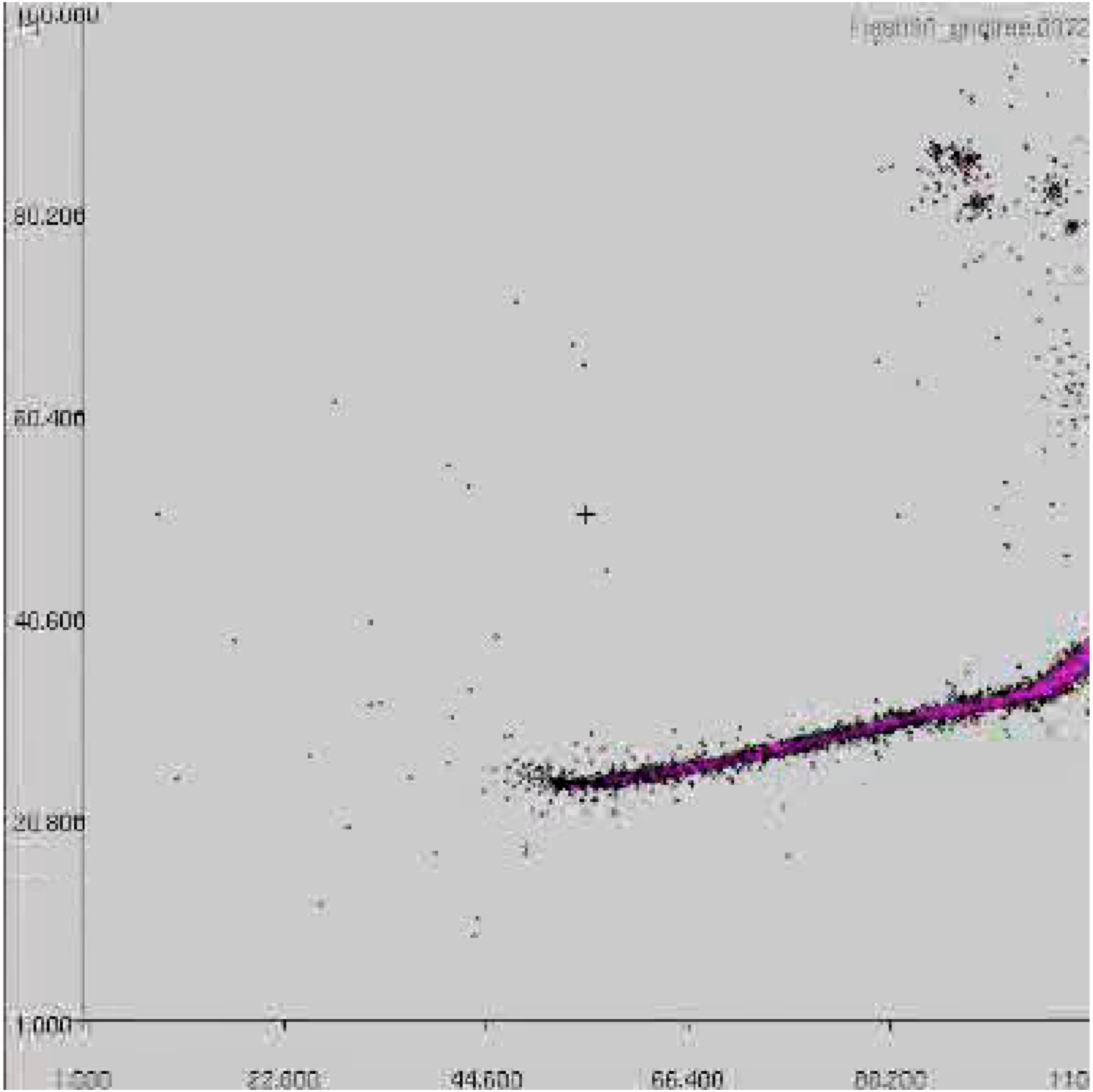


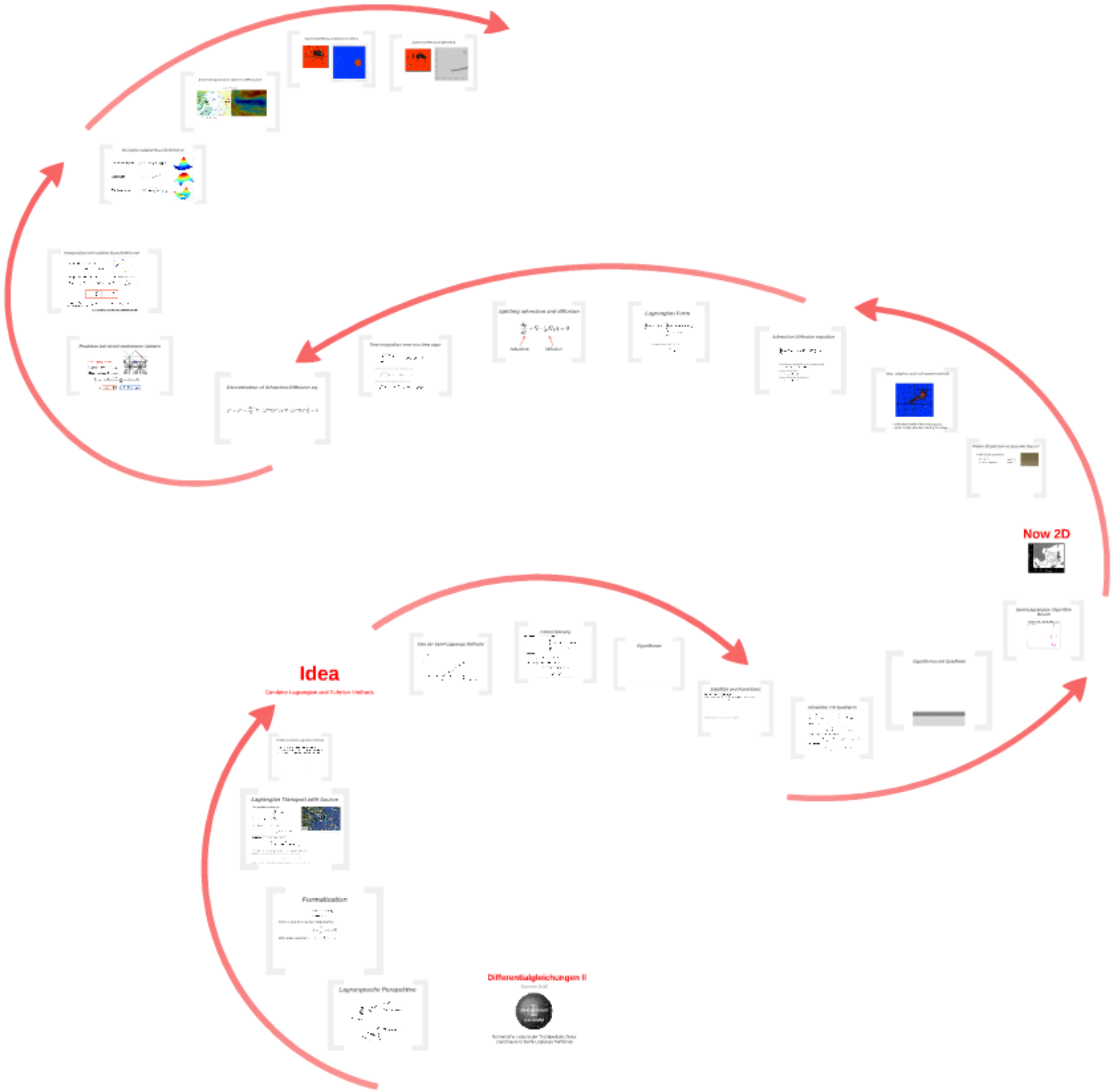

PLOT: tracer variable
(c) Jörn Behrens, 1998

Spurenstofftransport (gitterfrei)






100.000
80.200
60.400
40.600
20.800
1.000
1.000
22.800
44.600
66.400
88.200
110



Idea
Combine Lagrangian and Eulerian Methods
Formalization
Now 2D
Differentialgleichungen II