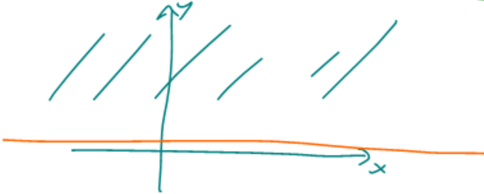
Lösungsmethoden

$$z'' = Az \qquad z(z) = \begin{cases} z_0 \sin\sqrt{-A}\,z + z_1 \cos\sqrt{-A}\,z & A<0 \\ z_0 + z_1 z & A=0 \\ z_0 \sinh\sqrt{A}\,z + z_1 \cosh\sqrt{A}\,z & A>0 \\ \left(\tilde{z}_0 e^{\sqrt{A}z} + \tilde{z}_1 e^{-\sqrt{A}z}\right) & \end{cases}$$

$$u_{xx} + u_{yy} = 0 \qquad \mathbb{R} \times (0,\infty)$$

$$u(x,0) = u_0(x) = 0$$

$$u_y(x,0) = v_0(x) = \frac{1}{n}\sin(nx)$$



Produktansatz: $u(x,y) = X(x)\cdot Y(y)$

$$X''Y + XY'' = 0 \qquad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{nur } x\text{-abh.}} = \underbrace{-\frac{Y''(y)}{Y(y)}}_{\text{nur } y \text{ ab}} \stackrel{!}{=} \text{const} = A$$

$A=0$

$$X''=0 \qquad X = x_0 + x_1 x \qquad u(x,y) = (x_0+x_1 x)(y_0+y_1 y)$$

$$Y''=0 \qquad Y = y_0 + y_1 y$$

$$u(x,0) = (x_0+x_1 x)\, y_0 \overset{!}{=} 0 \quad \Rightarrow y_0 = 0$$

$$u_y(x,0) = (x_0+x_1 x)\, y_1 \overset{!}{\neq} \frac{1}{n}\sin(nx)$$

$A=\delta^2>0$

$$X''=AX \qquad X(x) = x_0 \sinh \delta x + x_1 \cosh \delta x$$

$$Y''=-AY \qquad Y(y) = y_0 \sin \delta y + y_1 \cos \delta y$$

$$u(x,y) = (x_0 \sinh \delta x + x_1 \cosh \delta x)(y_0 \sin \delta y + y_1 \cos \delta y)$$

$$u_y(x,y) = (\qquad\qquad)\,\delta\,(y_0 \cos \delta y - y_1 \sin \delta y)$$

$$u(x,0) = (x_0 \sinh \delta x + x_1 \cosh \delta x)\, y_1 \overset{!}{=} 0 \quad \Rightarrow y_1 = 0$$

$$u_y(x,0) = (\quad -''- \quad)\,\delta\, y_0 \overset{!}{\neq} \frac{1}{n}\sin nx$$

$A = -\delta^2 < 0$

$$\ldots \quad u(x,y) = (x_0 \sin\delta x + \cancel{x_1 \cos\delta x})(y_0 \sinh\delta y + \cancel{y_1 \cosh\delta y})$$

$$u_y(x,y) = (\qquad\qquad)\,\delta(y_0 \cosh\delta y + y_1 \sinh\delta y)$$

$$u(x,0) = (\qquad\qquad)\, y_1 = 0 \quad \Rightarrow \quad y_1 = 0$$

$$u_y(x,0) = (x_0 \sin\delta x + x_1 \cos\delta x)\,\delta y_0 = \frac{1}{n}\sin nx \quad \Rightarrow \quad x_1 = 0, \quad \delta = n$$

$$x_0 \delta y_0 = \frac{1}{n} \Rightarrow x_0 y_0 = \frac{1}{n^2}$$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{n^2}\sin nx \sinh ny}$$

ad Wellengleichung $(c=1)$ "t" $\hat{=}$ "y"

$$-u_{xx} + u_{tt} = 0$$

$u(x,0) = u_0 =$ beschr.

$v(x,0) = v_0 = 0$

Produkt Ansatz

$u(x,t) = X(x)\,T(t)$

$$\ldots \quad \frac{X''}{X} = \frac{T''}{T} = \text{const} = A$$

$A = 0 \Rightarrow$ lineare Fkt $\Rightarrow$ unbeschr. (ausser const)

$A > 0 \Rightarrow x_0 \sinh\sqrt{A}x + x_1 \cosh\sqrt{A}x$ unbeschr.

z.B. $u(x,0) = \sin x$

$A = -\delta^2 < 0$

$$u(x,t) = (x_0 \sin \delta x + x_1 \cos \delta x)(t_0 \sin \delta t + t_1 \cos \delta t)$$

$$u_t(x,t) = (\qquad)\,\delta\,(t_0 \cos \delta t - t_1 \sin \delta t)$$

$$u(x,0) = (\qquad)\, t_1 \overset{!}{=} \sin x \Rightarrow x_1 = 0,\ \delta = 1,\ x_0 t_1 = 1$$

$$u_t(x,0) = (\qquad)\,\delta t_0 \overset{!}{=} 0 \Rightarrow t_0 = 0$$

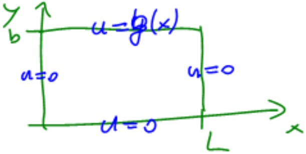
$$\Rightarrow u(x,t) = \sin x \cos t = \tfrac{1}{2}\left(\sin(x+t) + \sin(x-t)\right)$$

d'Alembert $u(x,t) = \frac{1}{2}\left(\sin(x+t) + \sin(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \cancel{v_0(\xi)\,d\xi}$

Ja, Produktansatz führt für sehr spezielle AB zum Ziel (= d'Alembert)

Zurück zu

$$\boxed{\begin{aligned} & u_{xx} + u_{yy} = 0 \\ & u(0,y) = u(L,y) = 0 \\ & u(x,0) = 0 \\ & u(x,b) = g(x) \end{aligned}}$$



Versuch: Produktansatz

$$u(x,y) = X(x)\, Y(y)$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = A$$

$A = 0$ $\quad X(x) = x_0 + x_1 x$ $\quad$ kann nicht bei $x=0$, $x=L$ verschwinden

$A = \delta^2 > 0$ $\quad u(x,y) = (x_0 \sinh\delta x + x_1 \cosh\delta x)(y_0 \sin\delta y + y_1 \cos\delta y)$

$u(0,y) = \quad x_1 \quad (\quad) \overset{!}{=} 0 \Rightarrow x_1 = 0$

$u(L,y) = (x_0 \sinh\delta L \quad)(\quad) \overset{!}{=} 0 \Rightarrow x_0 = 0$

$\Rightarrow u = 0$

$A = -\delta^2 < 0$ $\quad u(x,y) = (x_0 \sin\delta x + x_1 \cancel{\cos}\delta x)(y_0 \sinh\delta y + y_1 \cancel{\cosh}\delta y)$

$u(0,y) = \quad x_1 \quad (\quad) \overset{!}{=} 0 \Rightarrow x_1 = 0$

$u(L,y) = x_0 \sin\delta L \quad (\quad) \overset{!}{=} 0 \Rightarrow \delta L = k\pi$

$u(x,0) = x_0 \sin\delta x \quad (\quad y_1) \overset{!}{=} 0 \Rightarrow y_1 = 0$

$u(x,b) = x_0 \sin\delta x \quad (y_0 \sinh\delta b \quad) = g(x)$

scheint nur möglich

$\sinh\frac{k\pi b}{L}\, x_0 y_0 = C \quad \Leftarrow \quad g(x) = C \sin\frac{k\pi}{L} x$

$$\boxed{u(x,y) = C \,/\, \sinh\frac{k\pi}{L}b \cdot \sin\frac{k\pi}{L}x \,\sinh\frac{k\pi}{L}y}$$

Nun für spezielle RB ?

$u_{xx}+u_{yy}=0$ linear $\Rightarrow$ Falls $u_1=u_1(x,y)$ und $u_2=u_2(x,y)$ Lösungen

$\Rightarrow$ $u(x,y)=a_1u_1(x,y)+a_2u_2(x,y)$ Lsg

$$\boxed{\Delta u = \Delta(a_1u_1+a_2u_2) = a_1\Delta u_1 + a_2\Delta u_2 = 0}$$

d.h. falls $g(x)=\sum_{k=1}^{N} c_k \sin\frac{k\pi}{L}x$

$$\Rightarrow \boxed{u(x,y)=\sum_{k=1}^{N}\frac{c_k}{\sinh\frac{k\pi}{L}b}\sin\frac{k\pi}{L}x\,\sinh\frac{k\pi}{L}y}$$

d.h. falls $g(x)=\sum_{k=1}^{\infty} c_k \sin\frac{k\pi}{L}x$ Fourier-Sinusreihe $\leftarrow$ $c_k=\frac{2}{L}\int_0^L g(y)\sin\frac{k\pi}{L}y\,dy$

$$u(x,y)=\sum_{k=1}^{\infty}\frac{c_k}{\sinh\frac{k\pi}{L}b}\sin\frac{k\pi}{L}x\,\sinh\frac{k\pi}{L}y$$