Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

WS 2024/25

Differentialgleichungen 1 für Studierende der Ingenieurswissenschaften

Advice for exams

Claus R. Goetz

Universität Hamburg ● Fachbereich Mathematik

1. Exams Mathematics III:

Date, place, process, materials and tools

Date, place, process

Date: Tue., 04.03.25, 11:00 - 13:00 (Ana III and ODE I)

11:00 - 12:00 (only ODE I)

Place: Sporthalle Hamburg *(Alsterdorfer Sporthalle)*
Krochmannstr. 55
22297 Hamburg

If you are in this group.
you know that you are.

For **Ana III and ODE I** (will apply to most):

- You get two exams.
- You have 120 minutes, which you can spend as you prefer.
- In the end, both exams are treated as one big exam.

For **only ODE I**:

- You get one exam.
- You have 60 minutes.
- After 60 minutes you leave the hall.

More on the process

The hall will be separated into **three areas**:

- Only ODE I (mostly LuM)
- Ana III and ODE I (German)
- Ana III and ODE I (English)

When you're inside the hall:

- Take a seat, stay seated.
- We hand out the exams. You fill out the covers. You **must not open the exam** until you get the signal to do so. **Opening the exams early counts as an attempt to cheat!**

During the exam:

- We check the attendance. Please put a photo-ID and the cober of one exam easily visible on your desk.
- Unfortunately, we will have to briefly interrupt you when we're checking the cover of the exam you're working on.

Allowed materials

You **have to** bring:

- Something to write with (no pencils, no red ink), you do not need your own paper.
- An official photo-ID (passport, driver's licence, etc.)

You **can** bring:

- for each module 2 sheet DIN A4, written on front and back:
  - Mathe III (Ana III + ODE I): 4 sheets $\hat{=}$ 8 pages
  - Mathe III - ODE I: 2 sheets $\hat{=}$ 4 pages
- a non-smart watch that doesn't produce distracting sounds

You **must not** bring:

- smart phones, smart watches
- pocket calculators other other electronic devices
- textbooks, collections of formulas, etc., beyond what's indicated above

2. Klausur ODE I:

Vorkenntnisse, typische Themen, weitere Tipps

prerequisites

The focus of the exam will not be to test your skills when computing integrals. But you'll still have to do some calculations.

You will have to be able to do the following calculations routinely:

- (partial) derivatives
- elementary integrals
- Eigenvalues, eigenvectors and generalized eigenvectors
- solving linear systems
- basic arithmetic of complex numbers

Preliminary remarks

In the following we discuss some topics that have been typically occurred in previous exams. This should give you some orientation when preparing for the exam.

However:

- If a topic from the lecture / the exercises does not show up in these notes it **does not** follow that it cannot show up on the exam.
- We cannot repeat everything in these notes, the will only be some bullet point.

Tendency: Material from the work sheet are the basics, homework for deepening the understanding (Still important!)

Initial and boundary value problem

Points to remember:

- The **general solution** of an ODE contains free parameters.
- Through **initial conditions** (or boundary conditions) we can determine these parameters.
- A scalar problem of order m typically needs m conditions for the uniqueness of the solution.

Elementary techniques for first order scalar problems

You should be able to quickly identify the type of ODE.

Points to remember:

- We have learned different methods for solving ODES.
- Each of these can only be applied to certain types of ODEs.
- We can (often) solve linear and separable problems directly.
- Otherwise, we often try to transform our ODEs into a linear or separable problem.

Linear first order ODEs ($\rightarrow$ P2, HA1)

VERY IMPORTANT TOOL !

ODE (linear first order):

$$u'(t) = a(t)u(t) + b(t)$$

Example:

P2: A1, HA1

Ansatz:

$A(t)$: primitive of $a(t)$, $A(t) = \int a(t)\,\mathrm{d}t$,

$B^*(t)$: primitive of $\mathrm{e}^{-A(t)}b(t)$, $B^*(t) = \int \mathrm{e}^{-A(t)}b(t)\,\mathrm{d}t$.

Solution formula for the general solution:

$$u(t) = \mathrm{e}^{A(t)} \cdot [B^*(t) + C], \qquad C \in \mathbb{R}.$$

Solution formula for the IVP with $u(t_0) = y_0$:

$$u(t) = \mathrm{e}^{A(t)} \cdot \left[\int_{t_0}^{t} \mathrm{e}^{-A(s)}b(s)\,\mathrm{d}s + y_0\mathrm{e}^{-A(t_0)}\right].$$

Bernoulli-ODE

ODE (Bernoulli):

$$u'(t) \; = \; a(t)u(t) + b(t)u(t)^{\alpha}$$

$$\alpha \in \mathbb{R} \setminus \{0,1\}, \quad b \neq 0.$$

Example:

P2: A3

.

Ansatz: Substitution $y(t) = u(t)^{1-\alpha}$.

New ODE:

$$y'(t) = (1-\alpha)a(t)y(t) \; + \; (1-\alpha)b(t) \qquad \rightarrow \qquad \text{linear ODE}$$

Back-substitution: $u(t) = y(t)^{\frac{1}{1-\alpha}}$.

Separable ODE (→ P2, HA2, HA3)

VERY IMPORTANT TOOL !

ODE (separable):
$u'(t) = g(u) \cdot h(t)$

Example:
P2: A2, HA2

Ansatz: Check zeros of $g(u)$. For $g(u) \neq 0$: Separation of variables:

$$\int \frac{1}{g(u)}\, \mathrm{d}u = \int h(t)\, \mathrm{d}t.$$

ODE (self-similar):
$u'(t) = f\,(u(t)/t)$

Example:
HA3: A1

Ansatz: Substitution $y(t) = u(t)/t$. New ODE:

$$y'(t) = (f(y(t)) - y(t)) \cdot \frac{1}{t} \qquad \rightarrow \qquad \text{separable ODE}$$

Back-substitution: $u(t) = ty(t)$.

Exact ODEs $(\to$ P3)

ODE:

$$f(t,u) + g(t,u)u'(t) \;= 0$$

Example:

P3: A2, P3:A3

Ansatz:

1. Check for exactness: Is $f_u = g_t$?
2. If yes, compute potential $\Psi(t,u)$ by integrating.
3. Solve $\Psi(t,u) = C$ for u. (If possible.)

You also need this for Ana III

(Integrating factors are too complicated for the exam)

Linear, scalar, constant coefficients (→ P4, HA4)

VERY COMMON TOPIC FOR EXAMS.

ODE (linear, order m):

$$a_m u^{(m)} + a_{m-1}u^{(m-1)} + \cdots + a_1 u' + a_0 u = b, \qquad a_0, \ldots, a_m \in \mathbb{R}, \quad a_m \neq 0$$

Example:

P4: A1, A2, HA4:A2

Ansatz: For the homogeneous problem ($b = 0$):

- Characteristic polynomial: $p(\lambda) = a_m\lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_1\lambda + a_0 = 0$, roots λ_j, $1 \leq j \leq m$ (if necessary consider multiplicity).
- base solutions: $\mathrm{e}^{\lambda_j t}$ if λ_j is simple root, $\{\mathrm{e}^{\lambda_j t}, t\mathrm{e}^{\lambda_j t}\}$, if λ_j is double root
- fundamental system: Basis of the solution space (dimension m).
- Complex root only appear in pairs of conjugates. If necessary, find real solutions via $\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$.

Inhomogeneous problems / formulation as a System $(\to$ HA4)

General solution of the inhomogeneous problem ($b \neq 0$): $u = u_h + u_p$, with

u_h : general solution of the homogeneous problems (see above)

u_p : *one* particular solution of the inhomogeneous problems

we did not do this.

Find u_p by a special ansatz or (using variation of constants.)

Or: Write $y = (u, u', \ldots, u^{(m-1)})^\top$, determine the system $y' = Ay$ and use the tools from the next section ($\to$ HA4:A2).

we did not train this in this particular context, but it works like in all the other cases.

First order systems wth constant coefficients ($\rightarrow$ P4, P5, HA4, HA5)

ODE (linear $(m \times m)$-system):

EXTREMELY LARGE LIKELIHOOD TO BE ON THE EXAM.

$$u' = Au + b, \qquad A \in \mathbb{R}^{m\times m},\ b \in \mathbb{R}^m, \quad u : I \to \mathbb{R}^m$$

Example:

P4: A3, P5:A1 , HA5:A2

- Compute eigenvalues λ_j and corresponding eigenvectors $v^{[j]}$ of A.
- For simple eigenvalues: Base solution $\mathrm{e}^{\lambda_j t} v^{[j]}$.
- For double eigenvalues with geometric multiplicity one: Find generalized eigenvector $w^{[j]}$ via $(A - \lambda_j I)w^{[j]} = v^{[j]}$.
  Base solutions: $\mathrm{e}^{\lambda_j t} v^{[j]},\ \mathrm{e}^{\lambda_j t}\left(w^{[j]} + t v^{[j]}\right)$
- The set of all the base solutions is a **fundamental system**. The matrix the base solutions as columns is a **fundamental matrix**, $W(t)$.

More on linear first order systems with constant coefficients

For complex eigenvalues: $(\rightarrow$ P5:A2.(b), HA4:A1)

- complex eigenvalues and corresponding complex eigenvectors always appear in pairs of complex conjugates.
- for a complex eigenvalue λ_j with complex eigenvector v, we get real solutions via $\mathrm{Re}(\mathrm{e}^{\lambda t} v),\ \mathrm{Im}(\mathrm{e}^{\lambda t} v)$.

rather complicated

Inhomogeneous problem: $u' = Au + b(t)$:

- general solution: $u = u_h + u_p$.
- ansatz for a particular solution: $u_p(t) = W(t)k(t)$ with fundamental matrix W and coefficients k to be found.
- find k as solution of $W(t)k'(t) = b(t)$ $(\rightarrow$ P5:A2.(a)).
- Or: Find u_p using a special ansatz $(\rightarrow$ P5:A2.(b))

Stability for linear systems with constant coefficients $(\rightarrow$ P6)

For a system $u' = Au + b$ **equilibria** are solutions of $Au^* = -b$ $((u^*)' = 0)$.

It holds:

- If all eigenvalues of A are non-zero, there is exactly on equilibrium ($u^* = 0$ for $b = 0$).
- Stability (or instability, asymptotic stability) depends on the real parts of the eigenvalues.
- For $\mathrm{Re}(\lambda_j) = 0$ we have to consider the geometric multiplicity of λ_j.

If we've already computed the eigenvalues for the fundamental system, this one's easy!

The Laplace-transform $(\to$ P6)

ODE (linear, scalar, order m):

$$a_m u^{(m)} + a_{m-1} u^{(m-1)} + \cdots + a_1 u' + a_0 u = b, \qquad a_0, \ldots, a_m \in \mathbb{R}, \quad a_m \neq 0$$

Initial data for $u, u', \ldots, u^{(m-1)}$.

- Transform the ODE into an algebraic equation $(d_m s^m + \cdots + d_1 s + d_0) U(s) = B(s)$.
- Use rules for the correspondence of derivatives and initial (on the left), and find correspondences for the terms on the right in a table (hopefully).
- Solve for U.
- For the inverse transform: Find correspondences in a table. Use partial fractions and rules for shifts in the transform.

What will **not** be on the exam

The following will definitely not be on the exam:

- modelling problems
- drawings
- (complicated) mathematical proofs

Additional resources

Old exams:

- You can find old exams on the website for teaching export (link in Stud.IP). Most of the are in German only, though.
- Note that teachers in previous courses may have put their emphasis on different topics and may have used a different notation.
- The exam this year could also contain different types of problems.

Office hours / contact:

- In the weeks **17.02. - 21.02.** and **14.02. - 28.02.** we will offer additional office hours at UHH (dates will follow on the website for teaching export and in Stud.IP)
- Simple questions can be answered by mail (claus.goetz@uni-hamburg.de or DM in Stud.IP).