Differential Equations I for Students of Engineering Sciences

Homework 6

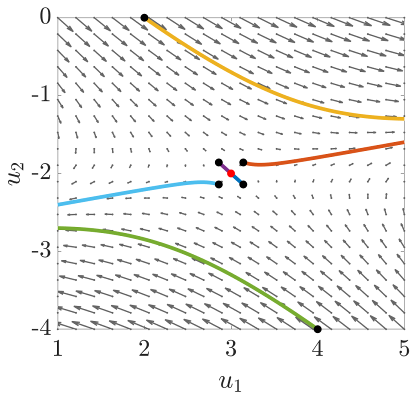
Problem 1: Draw a sketch of the vector fields defined by the following functions $F : \mathbb{R}^2 \to \mathbb{R}^2$. Inside these vector fields, sketch some trajectories of $u' = F(u)$, as well as the equilibria.

$$\text{(i)} \quad F(u) = \begin{pmatrix} u_1 + 5u_2 + 7 \\ u_1 - 3u_2 - 9 \end{pmatrix}, \qquad \text{(ii)} \quad F(u) = \begin{pmatrix} u_1 + u_2 \\ -u_1 + u_2 \end{pmatrix}$$

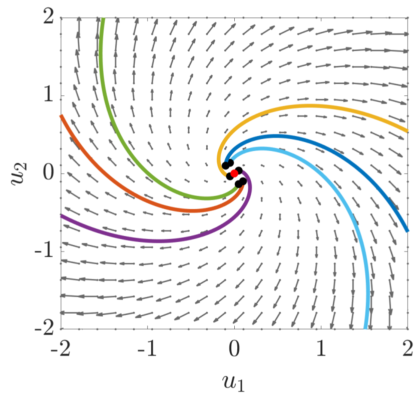
$$\text{(iii)} \quad F(u) = \begin{pmatrix} -u_1 - 2u_2 - 6 \\ 5u_1 + u_2 - 6 \end{pmatrix} \qquad \text{(iv)} \quad F(u) = \begin{pmatrix} -u_1 \\ -u_1 - u_2 \end{pmatrix}$$

Hint: We have found the equilibria for this on Work Sheet 6, Problem 1.

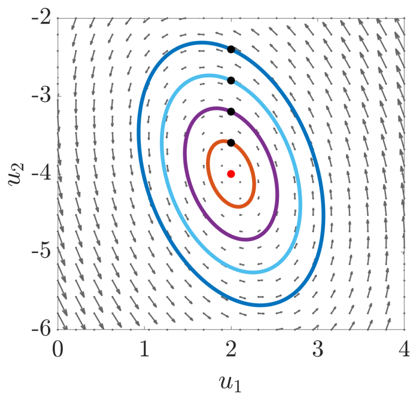
Solution.



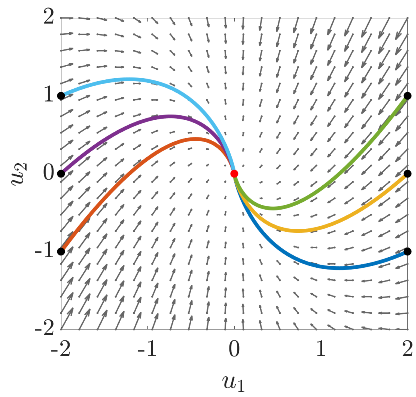
(i)



(ii)



(iii)



(iv)

Here, the red dots mark the equilibria and the black dots are the starting points of the trajectories at $t = 0$. We observe the following behaviour:

(i) There are trajectories that start near the equilibrium and and run into it. But there are also trajectories that start close to the equilibrium and then run off away from it. The equilibrium is a *saddle point* (unstable).

(ii) All trajectories that start close to equilibrium spiral outwards. This is an *unstable vortex.*

(iii) All trajectories run periodically around the equilibrium. This is a *circulation point* (stable, but not asymptotically stable).

(iv) All trajectories spiral into the equilibrium. This is a *stable vortex* (asymptotically stable).

Problem 2: We consider a model for an eco system: Assume that plants $u_1(t)$ are eaten by moose $u_2(t)$, and that the moose get eaten by wolves $u_3(t)$. We assume that the size of each population in isolation would follow a logistic growth model. Moreover, we assume that the rate of interaction between to species is proportional to the product of their population sizes. To keep things simple, we take all parameters in the model to be one.

We arrive at the system

$$\left.\begin{aligned} u_1' &= u_1(1-u_1) - u_1u_2, \\ u_2' &= u_2(1-u_2) + u_1u_2 - u_2u_3, \\ u_3' &= u_3(1-u_3) + u_2u_3. \end{aligned}\right\} \tag{$*$}$$

We write the system (**??**) as $u' = F(u)$ with $u = (u_1, u_2, u_3)^\top$.

(a) Compute the Jaobian $A(u) := JF(u)$ of F.

Solution.

$$A(u) = \begin{pmatrix} 1-2u_1-u_2 & -u_1 & 0 \\ u_2 & 1-2u_2+u_1-u_3 & -u_2 \\ 0 & u_3 & 1-2u_3+u_2 \end{pmatrix}.$$

(b) Show that for $u^* := (2/3,\ 1/3,\ 4/3)^\top$ the following holds: We have $F(u^*) = (0,0,0)^\top$ and $(0,0,0)^\top$ is an asymptotically stable equilibrium of the linearised system $u' = A(u^*)u$.

Reamrk: It can be shown that u^* is the only equilibrium of (**??**) in which all population sizes are positive.

Solution. The only equilibrium in which all population sizes are strictly positive is $u^* = (2/3,\ 1/3,\ 4/3)^\top$. That $F(u^*) = 0$ follows directly from plugging u^* into the function. In this point we find

$$A(u^*) = \begin{pmatrix} -2/3 & -2/3 & 0 \\ 1/3 & -1/3 & -1/3 \\ 0 & 4/3 & -4/3 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^3 + \frac{7}{3}\lambda^2 + \frac{20}{9}\lambda + \frac{8}{9}$$

with eigenvalues

$$\lambda_1 = -1, \quad \lambda_{2,3} = -\frac{2}{3} \pm \frac{2\mathrm{i}}{3}.$$

All eigenvalues have negative real part and the equilibrium is asymptotically stable.