Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 6

Problem 1: Draw a sketch of the vector fields defined by the following functions $F : \mathbb{R}^2 \to \mathbb{R}^2$. Inside these vector fields, sketch some trajectories of $u' = F(u)$, as well as the equilibria.

$$\text{(i)} \quad F(u) = \begin{pmatrix} u_1 + 5u_2 + 7 \\ u_1 - 3u_2 - 9 \end{pmatrix}, \qquad \text{(ii)} \quad F(u) = \begin{pmatrix} u_1 + u_2 \\ -u_1 + u_2 \end{pmatrix}$$

$$\text{(iii)} \quad F(u) = \begin{pmatrix} -u_1 - 2u_2 - 6 \\ 5u_1 + u_2 - 6 \end{pmatrix} \qquad \text{(iv)} \quad F(u) = \begin{pmatrix} -u_1 \\ -u_1 - u_2 \end{pmatrix}$$

Hint: We have found the equilibria for this on Work Sheet 6, Problem 1.

Problem 2: We consider a model for an eco system: Assume that plants $u_1(t)$ are eaten by moose $u_2(t)$, and that the moose get eaten by wolves $u_3(t)$. We assume that the size of each population in isolation would follow a logistic growth model. Moreover, we assume that the rate of interaction between to species is proportional to the product of their population sizes. To keep things simple, we take all parameters in the model to be one.

We arrive at the system

$$\left. \begin{aligned} u_1' &= u_1(1 - u_1) - u_1 u_2, \\ u_2' &= u_2(1 - u_2) + u_1 u_2 - u_2 u_3, \\ u_3' &= u_3(1 - u_3) + u_2 u_3. \end{aligned} \right\} \qquad (*)$$

We write the system $(*)$ as $u' = F(u)$ with $u = (u_1, u_2, u_3)^\top$.

(a) Compute the Jaobian $A(u) := JF(u)$ of F.

(b) Show that for $u^* := (2/3,\ 1/3,\ 4/3)^\top$ the following holds: We have $F(u^*) = (0, 0, 0)^\top$ and $(0, 0, 0)^\top$ is an asymptotically stable equilibrium of the linearised system $u' = A(u^*)u$.

Reamrk: It can be shown that u^* is the only equilibrium of $(*)$ in which all population sizes are positive.

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