Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. T. Schmidt, Dr. C. Goetz

Differential Equations I for Students of Engineering Sciences

Homework 5 - Solutions

Problem 1:

(a) Let $I \subset \mathbb{R}$ be an open interval and let $u_1,\ u_2,\ u_3 : I \to \mathbb{R}$ be twice continuously differentiable functions. The *Wronski determinant* is defined al

$$\mathrm{WD}(t) := \det \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{pmatrix}.$$

Proof the following: If u_1, u_2, u_3 are linearly dependent, we have $\mathrm{WD}(t) = 0$ for all $t \in I$. If, on the other hand, $\mathrm{WD}(t_0) \neq 0$ for some $t_0 \in I$, then u_1, u_2, u_3 are linearly independent.

Hint: Recall that the functions $u_1,\ u_2,\ u_3$ are linearly dependent, if there exists $(c_1, c_2, c_3)^\top \neq (0,0,0)^\top$, such that $c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) = 0$ for all $t \in I$.

(b) Show that the functions

$$u_1(t) = 1, \qquad u_2(t) = \mathrm{e}^{-t}\cos(t), \qquad u_3(t) = \mathrm{e}^{-t}\sin(t)$$

are linearly independent on $I = \mathbb{R}$.

(c) Find an equations of the form

$$a_3 u''' + a_2 u'' + a_1 u' + a_0 u = 0$$

with $a_0, \ldots, a_3 \in \mathbb{R}$, such that $M = \{1, \mathrm{e}^{-t}\cos(t),\ \mathrm{e}^{-t}\sin(t)\}$ is a fundamental system for that equation.

Solution.

(a) Let u_1, u_2, u_3 be linearly dependent. Then there exists a $c = (c_1, c_2, c_3)^\top \in \mathbb{R}^3 \setminus \{0\}$ with

$$c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) = 0 \qquad \text{for all } t \in I.$$

Moreover, then also it holds that

$$\begin{aligned} c_1 u_1'(t) + c_2 u_2'(t) + c_3 u_3'(t) = 0 \qquad &\text{for all } t \in I, \\ c_1 u_1''(t) + c_2 u_2''(t) + c_3 u_3''(t) = 0 \qquad &\text{for all } t \in I. \end{aligned}$$

Thus, the linear system of equations

$$\begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a non-trivial solution for each $t \in \mathbb{R}$. But that means that

$$\det \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{pmatrix} = 0, \qquad \text{für alle } t \in I.$$

From the negation of this statement we get: If $\mathrm{WD}(t) = 0$ *does not* hold *for all* $t \in I$, i.e. if $\mathrm{WD}(t_0) \neq 0$ holds *for some* $t_0 \in I$, then u_1, u_2, u_3 *are not* linearly dependent, i.e. they are linearly independent.

Note that in general this *does not* mean, that for an arbitrary set of three linearly independent functions we have $\mathrm{WD}(t) \neq 0$ *for all* $t \in I$. However, one can show: If in addition to being linearly independent u_1, u_2, u_3 also solve a linear, homogeneous differential equation of the form

$$a_3 u''' + a_2 u'' + a_1 u' + a_0 u = 0$$

with $a_0, \ldots, a_3 \in \mathbb{R}$, (in which case u_1, u_2, u_3 form a fundamental system for that equation), then indeed we have $\mathrm{WD}(t) \neq 0$ for all $t \in I$.

(b) We compute

$$u_1'(t) = 0, \qquad u_2'(t) = -\mathrm{e}^{-t}(\cos(t) + \sin(t)), \qquad u_3'(t) = \mathrm{e}^{-t}(\cos(t) - \sin(t)),$$

and

$$u_1''(t) = 0, \qquad u_2'' = 2\mathrm{e}^{-t}\sin(t), \qquad u_3'' = -2\mathrm{e}^{-t}\cos(t).$$

With that we have for all $t \in \mathbb{R}$:

$$\mathrm{WD}(t) = \det \begin{pmatrix} 1 & \mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t) \\ 0 & -\mathrm{e}^{-t}(\cos(t) + \sin(t)) & \mathrm{e}^{-t}(\cos(t) - \sin(t)) \\ 0 & 2\mathrm{e}^{-t}\sin(t) & -2\mathrm{e}^{-t}\cos(t) \end{pmatrix}.$$

Expanding along the first column we find

$$\begin{aligned} \mathrm{WD}(t) = {} & \left[-\mathrm{e}^{-t}(\cos(t) + \sin(t))\right] \cdot \left[-2\mathrm{e}^{-t}\cos(t)\right] - \left[\mathrm{e}^{-t}(\cos(t) - \sin(t))\right] \cdot \left[2\mathrm{e}^{-t}\sin(t)\right] \\ &= 2\mathrm{e}^{-2t}\left[\cos^2(t) + \sin(t)\cos(t)\right] - 2\mathrm{e}^{-2t}\left[\sin(t)\cos(t) - \sin^2(t)\right] \\ &= 2\mathrm{e}^{-2t}(\sin^2(t) + \cos^2(t)) = 2\mathrm{e}^{-2t} > 0 \qquad \text{für alle } t \in \mathbb{R}. \end{aligned}$$

therefore, u_1, u_2, u_3 linearly independent.

(c) The functions have the form

$$1 = \mathrm{e}^{0 \cdot t}, \qquad \mathrm{e}^{-t}\cos(t) = \mathrm{Re}\left(\mathrm{e}^{(-1+\mathrm{i})t}\right) \qquad \mathrm{e}^{-t}\sin(t) = \mathrm{Im}\left(\mathrm{e}^{(-1+\mathrm{i})t}\right).$$

That corresponds to the roots of the polynomial

$$p(\lambda) = \lambda \cdot (\lambda - (-1 + \mathrm{i})) \cdot (\lambda - (-1 - \mathrm{i})) = \lambda^3 + 2\lambda^2 + 2\lambda.$$

The equation

$$u''' + 2u'' + 2u' = 0.$$

has the above polynomial as its characteristic polynomial.

Problem 2: Find the eigenvalues of the following matrix. Determine the corresponding eigenvectors, and, if necessary, the generalized eigenvectors.

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$$

Solution.

Eigenvalues:

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{pmatrix} = (1-\lambda)\det \begin{pmatrix} -5-\lambda & 6 \\ -3 & 4-\lambda \end{pmatrix} \\ &= (1-\lambda)((-5-\lambda)(4-\lambda)+18) = (1-\lambda)(\lambda^2+\lambda-2) \\ &= -(\lambda-1)^2(\lambda+2). \end{aligned}$$

We have $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 1$ with algebraic multiplicity two.

Eigenvector $v^{[1]}$ for $\lambda_1 = -2$:

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & -3 & 6 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can choose, e. g.,

$$v^{[1]} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Eigenvectors for $\lambda_2 = 1$:

$$\left(\begin{array}{ccc|c} o & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We see that any solution of this has to have the form

$$\begin{pmatrix} a \\ b \\ b \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

With

$$v^{[2]} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v^{[3]} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

we have found two linearly independent eigenvectors. The geometric multiplicity therefore is equal to the algebraic multiplicity and we do not need generalized eigenvectors.