Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 5

Problem 1:

(a) Let $I \subset \mathbb{R}$ be an open interval and let $u_1,\ u_2,\ u_3 : I \to \mathbb{R}$ be twice continuously differentiable functions. The *Wronski determinant* is defined al

$$\mathrm{WD}(t) := \det \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{pmatrix}.$$

Proof the following: If u_1, u_2, u_3 are linearly dependent, we have $\mathrm{WD}(t) = 0$ for all $t \in I$. If, on the other hand, $\mathrm{WD}(t_0) \neq 0$ for some $t_0 \in I$, then u_1, u_2, u_3 are linearly independent.

Hint: Recall that the functions $u_1,\ u_2,\ u_3$ are linearly dependent, if there exists $(c_1, c_2, c_3)^\top \neq (0,0,0)^\top$, such that $c_1u_1(t) + c_2u_2(t) + c_3u_3(t) = 0$ for all $t \in I$.

(b) Show that the functions

$$u_1(t) = 1, \qquad u_2(t) = \mathrm{e}^{-t}\cos(t), \qquad u_3(t) = \mathrm{e}^{-t}\sin(t)$$

are linearly independent on $I = \mathbb{R}$.

(c) Find an equations of the form

$$a_3u''' + a_2u'' + a_1u' + a_0u = 0$$

with $a_0, \dots, a_3 \in \mathbb{R}$, such that $M = \{1, \mathrm{e}^{-t}\cos(t),\ \mathrm{e}^{-t}\sin(t)\}$ is a fundamental system for that equation.

Problem 2: Find the eigenvalues of the following matrix. Determine the corresponding eigenvectors, and, if necessary, the generalized eigenvectors.

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$$

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