Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 4 - Solutions

Problem 1: We consider a linear *predator-prey-model*. Suppose there are two species of fish: The predators u_1 and the prey u_2, where we assume that the predators feed on the prey.

(a) We describe the evolution in time of the two populations by

$$\begin{aligned} u_1'(t) &= -a_{1,1}u_1(t) + a_{1,2}u_2(t) + b_1, \\ u_2'(t) &= -a_{2,1}u_1(t) + a_{2,2}u_2(t) + b_2, \end{aligned}$$

where $a_{i,j} > 0$ for $1 \le i, j \le 2$ and $b_1, b_2 \in \mathbb{R}$ are given real parameters.

What do the terms in these equations mean? What processes are described by them?

Solution: The term $a_{1,2}u_2$ with $a_{1,2} > 0$ in the first equation means the population of predators grows proportionally to the supply in food. We can assume that the predators would go extinct without the other population to prey on, which is reflected by the term $-a_{1,1}u_1(t)$ with $a_{1,1} > 0$.

In the second equation we see that the population of the prey grows proportionally to its size ($a_{2,2}u_2$, with $a_{2,2} > 0$) and shrinks proportionally to the size of the predator population ($-a_{2,1}u_1$ with $a_{2,1} > 0$).

The terms b_1 and b_2 can describe exogenous effects, such as fishing or migration of fish from outside.

(b) Let

$$a_{1,1} = a_{2,1} = a_{2,2} = \frac{1}{3}, \quad a_{1,2} = \frac{2}{3}, \quad b_1 = b_2 = 0, \qquad A := \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Show that the functions

$$w_1(t) = \mathrm{e}^{\frac{\mathrm{i}}{3}t} \begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix}, \qquad w_2(t) = \mathrm{e}^{-\frac{\mathrm{i}}{3}t} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix},$$

form a *complex* fundamental system, and that

$$q_1(t) = \begin{pmatrix} \sin(t/3) + \cos(t/3) \\ \cos(t/3) \end{pmatrix}, \qquad q_2(t) = \begin{pmatrix} \sin(t/3) - \cos(t/3) \\ \sin(t/3) \end{pmatrix},$$

form a *real* fundamental system of the homogenous systems $u' = Au,\ u = (u_1, u_2)^\top$.

Solution: The eigenvalues are the roots of the characteristic polynomial:

$$0 = \det\begin{pmatrix} -\frac{1}{3}-\lambda & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3}-\lambda \end{pmatrix} = -\left(\frac{1}{3}+\lambda\right)\left(\frac{1}{3}-\lambda\right)+\frac{2}{9}$$
$$= \lambda^2 + \frac{1}{9} \quad \Rightarrow \quad \lambda_{1,2} = \pm\frac{\mathrm{i}}{3}.$$

The corresponding eigenvectors, $Av_k = \lambda_k v_k,\ k = 1, 2$, are

$$v_1 = \begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix}.$$

Therefore, $w_1(t) = \mathrm{e}^{\lambda_1 t}v_1,\ w_2(t) = \mathrm{e}^{\lambda_2 t}v_2$ form a complex fundamental system.

From this we can construct a real fundamental system by taking $\mathrm{R}e(w_1),\ \mathrm{I}m(w_1)$. By the Euler identity we have

$$\mathrm{e}^{\mathrm{i}\frac{t}{3}} = \cos(t/3) + \mathrm{i}\sin(t/3)$$

and thus

$$\begin{aligned}\mathrm{e}^{\frac{\mathrm{i}}{3}t}\begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix} &= (\cos(t/3) + \mathrm{i}\sin(t/3))\begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t/3) + \mathrm{i}\sin(t/3) - \mathrm{i}\cos(t/3) + \sin(t/3) \\ \cos(t/3) + \mathrm{i}\sin(t/3) \end{pmatrix} \\ &= \begin{pmatrix} \sin(t/3) + \cos(t/3) \\ \cos(t/3) \end{pmatrix} + \mathrm{i}\begin{pmatrix} \sin(t/3) - \cos(t/3) \\ \sin(t/3) \end{pmatrix}.\end{aligned}$$

We see that $q_1 = \mathrm{Re}(w_1),\ q_2 = \mathrm{I}m(w_1)$ and these function form a real fundamental system.

(c) Solve the *homogenous* initial value problem $u' = Au,\ u(0) = (4, 8)^\top$. Are u_1, u_2 positive for all $t > 0$?

Solution: We can write the solution in terms of the basis solutions:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = c_1\begin{pmatrix} \sin(t/3) + \cos(t/3) \\ \cos(t/3) \end{pmatrix} + c_2\begin{pmatrix} \sin(t/3) - \cos(t/3) \\ \sin(t/3) \end{pmatrix},$$

where the coefficients $c_1,\ c_2 \in \mathbb{R}$ are determined by the initial data. We have

$$\begin{pmatrix} 4 \\ 8 \end{pmatrix} = \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

From that it follows that $c_1 = 8$ and $4 = 8 - c_2$, so $c_2 = 4$. The solution of the initial value problem is then given by

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 12\sin(t/3) + 4\cos(t/3) \\ 4\sin(t/3) + 8\cos(t/3) \end{pmatrix}.$$

We have $u_2(t) = 8\cos(t/3) + 4\sin(t/3)$. Roughly speaking: The $8\cos(t/3)$ part can become negative and then the $4\sin(t/3)$ part is too small to get a positive solution. Analogously, also u_1 admits negative values.

More precisely: For $t_* = 3(\arctan(-2) + \pi) \approx 6.103$ it holds $u_2(t_*) = 0$ and for $t \in (t_*,\ t_* + 3\pi)$ we have $u_2(t) < 0$. For the linear homogenous 2×2 model, we cannot get periodic solutions that are positive for all $t > 0$.

(d) Solve the *inhomogeneous* initial value problem $u' = Au + b,\ u(0) = (4, 8)^\top$ with $b = (-4, 2)^\top$. Sketch the solution for $t \in [0, 12\pi]$. Describe the qualitative behaviour of the solution.

Hint: The inhomogeneous problem admits a constant particular solution.

Solution: For a constant particular solution u_p it follows that

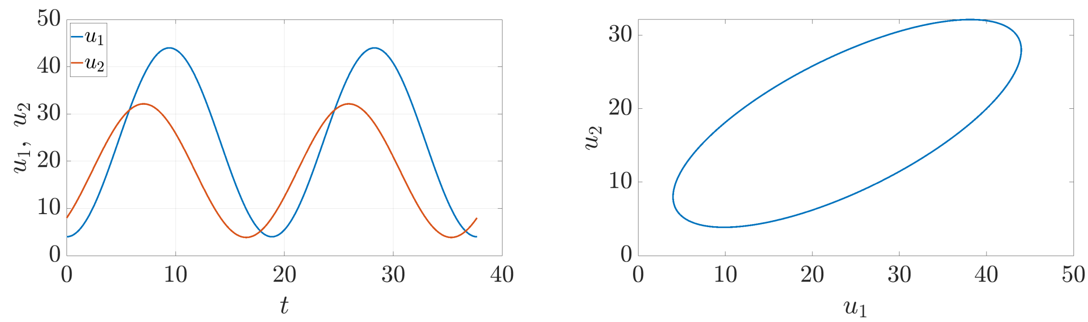
$$0 = u_p' = Au_p + b \qquad \Rightarrow \qquad u_p = -A^{-1}b \quad \Rightarrow \quad u_p = -\begin{pmatrix} 3 & -6 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 24 \\ 18 \end{pmatrix}.$$

The general solution of the inhomogeneous problem is $u = c_1 q_1 + c_2 q_2 + u_p$ and we get from the initial data that

$$u_1(0) = 4 = c_1 - c_2 + 24, \qquad u_2(0) = 8 = c_1 + 18 \quad \Rightarrow \quad c_1 = -10, \qquad c_2 = 10.$$

The solution of the initial value problem is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 24 - 20\cos(t/3) \\ 18 + 10\sin(t/3) - 10\cos(t/3) \end{pmatrix}.$$



At first, both populations grow until there are so many predators that the prey population starts to decline. The predator population keeps growing until there is no longer enough food, so it starts to shrink. If the number of predators is sufficiently small the prey recover and their population starts growing again. This, with some delay, leads to a growing population of predators. This process repeats periodically.

Problem 2: Consider the differential equation

$$u''' - 4u'' - 20u' + 48u = 0.$$

(a) Determine the general solution of this equation.

Solution: Characteristic polynomial:

$$p(\lambda) = \lambda^3 - 4\lambda^2 - 20\lambda + 48 = (\lambda+4)(\lambda-2)(\lambda-6) \stackrel{!}{=} 0\,.$$

The roots $\lambda_1 = -4,\ \lambda_2 = 2,\ \lambda_3 = 6$ lead to the fundamental system $u_1(t) = e^{-4t}$, $u_2(t) = e^{2t}$, $u_3(t) = e^{6t}$. Then, the general solution is

$$u(t) = c_1 e^{-4t} + c_2 e^{2t} + c_3 e^{6t} \quad \text{with} \quad c_1, c_2, c_3 \in \mathbb{R}\,.$$

(b) Write the equation as a first order system. For this system, compute the eigenvalues and eigenvectors and determine a fundamental matrix.

Solution: With $(u_0, u_1, u_2)^\top := (u, u', u'')^\top$ we can write:

$$\begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}' = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & 20 & 4 \end{pmatrix}}_{:=A} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

Expanding along the first row we find

$$\det(A - \lambda I) = -\lambda(-\lambda(4-\lambda) - 20) - 48 = -\lambda^3 + 4\lambda^2 + 20\lambda - 48 = -p(\lambda).$$

Therefore, the eigenvalues are $\lambda_1 = -4,\ \lambda_2 = 2,\ \lambda_3 = 6$. The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ -4 \\ 16 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 6 \\ 36 \end{pmatrix}.$$

From that we can construct a fundamental matrix as

$$W(t) = \begin{pmatrix} \mathrm{e}^{-4t} & \mathrm{e}^{2t} & \mathrm{e}^{6t} \\ -4\mathrm{e}^{-4t} & 2\mathrm{e}^{2t} & 6\mathrm{e}^{6t} \\ 16\mathrm{e}^{-4t} & 4\mathrm{e}^{2t} & 36\mathrm{e}^{6t} \end{pmatrix}.$$

In this case we can also derive the fundamental matrix directly from the fundamental system in part (a): In the first row we have the basis solution, in the second row their first derivatives and in the third row their second derivatives.