Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 4

Problem 1: We consider a linear *predator-prey-model*. Suppose there are two species of fish: The predators u_1 and the prey u_2, where we assume that the predators feed on the prey.

(a) We describe the evolution in time of the two populations by

$$\begin{aligned} u_1'(t) &= -a_{1,1}u_1(t) + a_{1,2}u_2(t) + b_1, \\ u_2'(t) &= -a_{2,1}u_1(t) + a_{2,2}u_2(t) + b_2, \end{aligned}$$

where $a_{i,j} > 0$ for $1 \le i, j \le 2$ and $b_1, b_2 \in \mathbb{R}$ are given real parameters.

What do the terms in these equations mean? What processes are described by them?

(b) Let

$$a_{1,1} = a_{2,1} = a_{2,2} = \frac{1}{3}, \quad a_{1,2} = \frac{2}{3}, \quad b_1 = b_2 = 0, \qquad A := \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Show that the functions

$$w_1(t) = \mathrm{e}^{\frac{\mathrm{i}}{3}t} \begin{pmatrix} 1-\mathrm{i} \\ 1 \end{pmatrix}, \qquad w_2(t) = \mathrm{e}^{-\frac{\mathrm{i}}{3}t} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix},$$

form a *complex* fundamental system, and that

$$q_1(t) = \begin{pmatrix} \sin(t/3) + \cos(t/3) \\ \cos(t/3) \end{pmatrix}, \qquad q_2(t) = \begin{pmatrix} \sin(t/3) - \cos(t/3) \\ \sin(t/3) \end{pmatrix},$$

form a *real* fundamental system of the homogenous systems $u' = Au$, $u = (u_1, u_2)^\top$.

(c) Solve the *homogenous* initial value problem $u' = Au$, $u(0) = (4, 8)^\top$. Are u_1, u_2 positive for all $t > 0$?

(d) Solve the *inhomogeneous* initial value problem $u' = Au + b$, $u(0) = (4, 8)^\top$ with $b = (-4, 2)^\top$. Sketch the solution for $t \in [0, 12\pi]$. Describe the qualitative behaviour of the solution.

Hint: The inhomogeneous problem admits a constant particular solution.

Problem 2: Consider the differential equation

$$u''' - 4u'' - 20u' + 48u = 0.$$

(a) Determine the general solution of this equation.

(b) Write the equation as a first order system. For this system, compute the eigenvalues and eigenvectors and determine a fundamental matrix.

Submit by: 13. December 2024