Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. T. Schmidt, Dr. C. Goetz

Differential Equations I for Students of Engineering Sciences

Homework 2 - Solutions

Problem 1: Consider a cylindrical water barrel, such that the water is flowing out through a circular hole on the bottom of the barrel. We denote the water height inside the barrel by h. Under simplified assumptions (no friction, no turbulence, ...) we can describe the evolution of h by the following differential equation:

$$h'(t) = -K\sqrt{h(t)}. \tag{1}$$

Here, $K > 0$ is a constant that depends on the diameter of the barrel, the diameter of the hole, and the gravity constant.

(a) Let $K > 0$ and $h_0 > 0$ be given. Solve the initial value problem

$$h'(t) = -K\sqrt{h(t)}, \qquad h(0) = h_0,$$

by separation of variables.

Compute the time t_* for which $h(t_*) = 0$.

Solution: Using separation of variables we compute

$$\begin{aligned} \int \frac{1}{\sqrt{h}} dh &= \int -K \, dt \\ \Rightarrow \quad 2\sqrt{h} &= -Kt + c, \qquad \text{with } c \in \mathbb{R}, \\ \Rightarrow \quad h(t) &= \frac{1}{4}(c - Kt)^2. \end{aligned}$$

From the initial condition we get

$$h(0) = h_0 = \frac{c^2}{4},$$

and thus with $c = 2\sqrt{h_0}$ we get

$$h(t) = \left(\sqrt{h_0} - \frac{K}{2}t\right)^2$$

as the solution of the initial value problem. From this it follows that

$$t_* = \frac{2\sqrt{h_0}}{K}.$$

(b) Is the solution of the initial value problem from part (a) defined for all $t > t_*$?

Solution: The function h is defined for all $t > 0$. However, equation (1) requires $h' \le 0$. Since h grows for $t > t_*$, it does not satisfy the differential equation for $t > t_*$. This makes sense, as a growing water height would not be physically reasonable.

(c) Let h be the solution of the initial value problem from part (a) and $h(t_*) = 0$. Show that the function $\tilde{h} : [0, \infty) \longrightarrow [0, \infty)$,

$$\tilde{h}(t) = \begin{cases} h(t) & \text{for } \ 0 \le t \le t_*, \\ 0 & \text{for } \ t > t_*, \end{cases}$$

is a solution of the initial value problem. In particular, show that $\tilde{h}$ is differentiable for all $t > 0$.

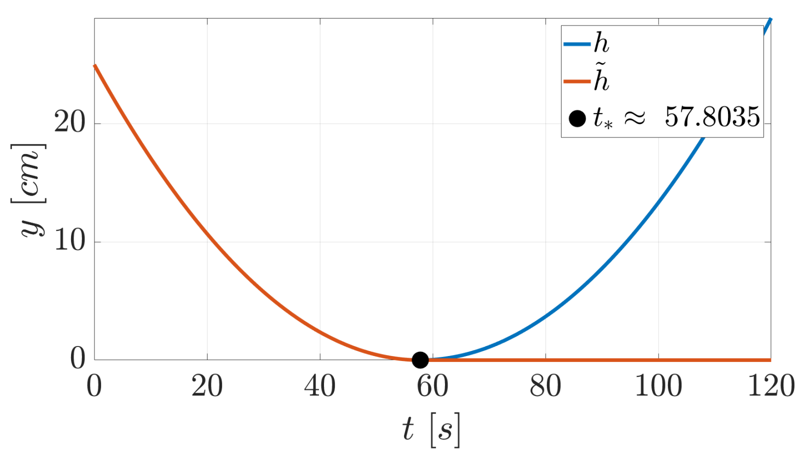
Solution: The zero function is a solution of (1), so $\tilde{h}$ is piecewise a solution. We have to check whether the differential equation is also satisfied in $t = t_*$. The function $\tilde{h}$ is continuous and piecewise differentiable. We compute

$$\tilde{h}'(t) = \begin{cases} -K\left(\sqrt{h_0} - \frac{K}{2}t\right) & \text{for } \ 0 \le t < t_*, \\ 0 & \text{for } \ t > t_*. \end{cases}$$

Since

$$\lim_{t \nearrow t_*} \left[-K \left(\sqrt{h_0} - \frac{K}{2} t \right) \right] = 0$$

it follows that $\tilde{h}$ is differentiable also in $t = t_*$ with $\tilde{h}(t_*) = 0$. Therefore, $\tilde{h}$ is a solution of (1). Because $\tilde{h}(0) = h(0) = h_0$, it also satisfies the initial condition.



Example:

$K = 0.173, \qquad h_0 = 25,$

corresponds to a barrel with diameter 16 *cm and a hole with diameter* 1 *cm.*

(d) Now we do not impose the initial condition $h(0) = h_0$, but rather $h(T) = 0$ for some given $T > 0$, i.e,

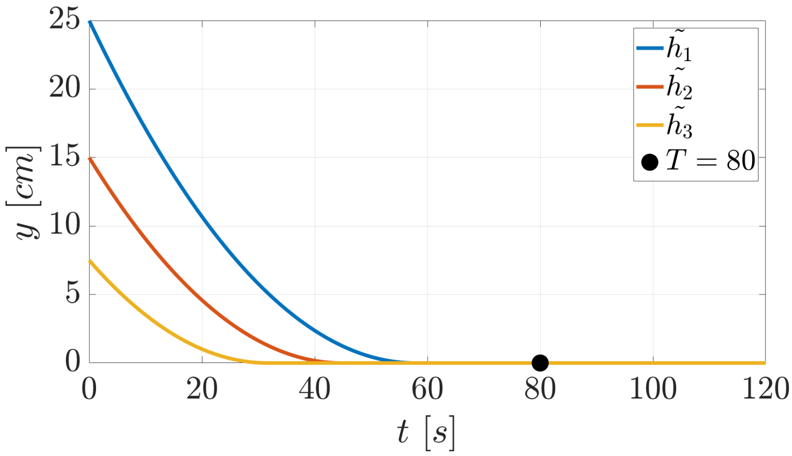
$$h'(t) = -K\sqrt{h(t)}, \qquad h(T) = 0. \tag{2}$$

How many solutions are there for this problem?

Lösung: The zero function is one solution. Moreover, all functions $\tilde{h}$ of the form from (c) are solutions if $t_* \leq T$. For each h_0 with

$$0 < h_0 \leq \left(\frac{K}{2}T\right)^2$$

the corresponding $\tilde{h}$ solves problem (2). We see that there are infinitely many solutions.



Problem (2) with

$$K = 0.173, \qquad T = 80.$$

The solution is not unique.

We can interpret this result in the following way: If we look inside the barrel at some time $t > 0$ and find that there is still water inside of it, we can tell how much water was inside the barrel before and much will be inside it later. If, on the other hand, we find the barrel to be empty at some time $T > 0$, we cannot say how much water was inside the barrel or since when it is empty.

We will later learn that the non-uniqueness of solution has to do with the fact that the function $f(h) = \sqrt{h}$ is *not* Lipschitz-continuous at $h = 0$.

Problem 2: Find a solution of the differential equation

$$y' - 6y + 3x^2y^2 = -\frac{2}{x^3} - \frac{3}{x^2}, \qquad x > 0,$$

by using the ansatz $y(x) = cx^\alpha$. That is, find suitable parameters $c, \alpha \in \mathbb{R}$, such that $y(x) = cx^\alpha$ solves the differential equation.

Solution: With this ansatz we get

$$-2x^{-3} - 3x^{-2} \;=\; y' - 6y + 3x^2y^2 \;=\; c\alpha x^{\alpha-1} - 6cx^\alpha + 3c^2x^{2\alpha+2}.$$

First, we choose the power α. On the left hand side we only have two powers: -2 and -3. On the right hand side, however, we have three powers: $\alpha - 1$, α, $2\alpha + 2$. This can only hold if two powers on the right hand side are equal. Since $\alpha \neq \alpha - 1$, this means either $\alpha = 2\alpha + 2$, or $\alpha - 1 = 2\alpha + 2$.

In the first case we find

$$\alpha = 2\alpha + 2 \qquad \Rightarrow \qquad \alpha = -2, \quad \alpha - 1 = -3,$$

which matches the powers on the left hand side. The other case leads to $\alpha = -3$, $\alpha - 1 = -4$, which does not provide a solution.

With $\alpha = -2$ we have

$$-2x^{-3} - 3x^{-2} \;=\; -2cx^{-3} - 6cx^{-2} + 3c^2x^{-2} \;=\; -2cx^{-3} + (3c^2 - 6c)x^{-2}$$

and by comparing the coefficients, it follows that $c = 1$, and so $y(x) = x^{-2}$.