Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 2

Problem 1: Consider a cylindrical water barrel, such that the water is flowing out through a circular hole on the bottom of the barrel. We denote the water height inside the barrel by h. Under simplified assumptions (no friction, no turbulence, ...) we can describe the evolution of h by the following differential equation:

$$h'(t) = -K\sqrt{h(t)}. \tag{1}$$

Here, $K > 0$ is a constant that depends on the diameter of the barrel, the diameter of the hole, and the gravity constant.

(a) Let $K > 0$ and $h_0 > 0$ be given. Solve the initial value problem

$$h'(t) = -K\sqrt{h(t)}, \qquad h(0) = h_0,$$

by separation of variables.

Compute the time t_* for which $h(t_*) = 0$.

(b) Is the solution of the initial value problem from part (a) defined for all $t > t_*$?

(c) Let h be the solution of the initial value problem from part (a) and $h(t_*) = 0$. Show that the function $\tilde{h} : [0, \infty) \longrightarrow [0, \infty)$,

$$\tilde{h}(t) = \begin{cases} h(t) & \text{for } 0 \leq t \leq t_*, \\ 0 & \text{for } t > t_*, \end{cases}$$

is a solution of the initial value problem. In particular, show that $\tilde{h}$ is differentiable for all $t > 0$.

(d) Now we do not impose the initial condition $h(0) = h_0$, but rather $h(T) = 0$ for some given $T > 0$, i.e,

$$h'(t) = -K\sqrt{h(t)}, \qquad h(T) = 0. \tag{2}$$

How many solutions are there for this problem?

Problem 2: Find a solution of the differential equation

$$y' - 6y + 3x^2y^2 = -\frac{2}{x^3} - \frac{3}{x^2}, \qquad x > 0,$$

by using the ansatz $y(x) = cx^\alpha$. That is, find suitable parameters $c, \alpha \in \mathbb{R}$, such that $y(x) = cx^\alpha$ solves the differential equation.

Submit by: 15. November 2024