Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 1 - Solutions

We consider a population of fish in the sea, which we denote by $y(t)$ at time $t \geq 0$. We assume as first (at least for large times unrealistic) step that there is an unlimited amount of food and space available to the fish, and that they are not influenced by any external factors (predators, fishing, etc.). Then the number of fish only varies due to natural births and deaths.

We assume that there is a constant birth rate $m \in [0,1]$ and a constant death-rate $n \in [0,1]$, respectively. For a short time step $\Delta t > 0$ we can describe the evolution of the population by

$$y(t+\Delta t) = y(t) + \Delta t\,(m \cdot y(t) - n \cdot y(t))\,. \tag{1}$$

(a) Deduce from this a differential equation. Solve the corresponding initial value problem with initial value $y(0) = y_0 > 0$. Show that with the *reproduction rate*

$$r := m - n, \qquad r \in [-1,1],$$

for the solution y of the initial value problem it holds:

$$\lim_{t\to\infty} y(t) \;=\; \begin{cases} \infty & \text{for } r > 0, \\ y_0 & \text{for } r = 0, \\ 0 & \text{for } r < 0. \end{cases}$$

Solution. Under the assumption that y is differentiable, we obtain from (1):

$$\frac{y(t+\Delta t) - y(t)}{\Delta t} = (m-n)y(t).$$

Passing to the limit for $\Delta t \to 0$ it becomes

$$y'(t) = (m-n)y(t) = ry(t).$$

The initial value problem $y' = ry, y(0) = y_0 > 0$ has the solution

$$y(t) = y_0 e^{rt}$$

and the claim follows directly from the behavior of the exponential function.

In the following we suppose that an any time unit a constant number $k > 0$ of fish is caught. The corresponding initial value problem reads

$$\begin{cases} y'(t) = ry(t) - k & \text{for } t > 0, \\ y(0) = y_0 & \text{for } t = 0. \end{cases}$$

(b) Solve this initial value problem.

Solution. With $A(t) = rt$ and $b(t) = -k$ from the solution formula for initial value problems it returns:

$$\begin{aligned} y(t) &= \mathrm{e}^{rt}\left[\int_0^t \mathrm{e}^{-rs}(-k)\,\mathrm{d}s + y_0\right] = y_0\mathrm{e}^{rt} + \frac{k}{r}\mathrm{e}^{rt}\cdot \mathrm{e}^{-rs}\Big|_0^t \\ &= y_0\mathrm{e}^{rt} + \frac{k}{r}\mathrm{e}^{rt}\left(\mathrm{e}^{-rt} - 1\right) \\ &= \left(y_0 - \frac{k}{r}\right)\mathrm{e}^{rt} + \frac{k}{r}. \end{aligned}$$

(c) Determine, according to k, r and y_0, if the population increases or decreases. Can it happen that the population remains constant?

Could negative values of y occur? How could these be possibly interpreted?

Solution. For $r > 0$ and $k < ry_0$, $\left(y_0 - \frac{k}{r}\right)\mathrm{e}^{rt}$ is positive and increasing for $t \to \infty$ without an upper bound. This happens when the number of fish caught k is smaller than the reproduction of the original population (ry_0).

For $r > 0$ and $k > ry_0$, $\left(y_0 - \frac{k}{r}\right)\mathrm{e}^{rt}$ is negative and it converges monotonously to $-\infty$ for $t \to \infty$. Thus the population decreases, and for t large enough $y(t)$ gets negative. Purely mathematically, this is not an issue, but populations smaller than zero do not have a biological meaning, and the model no longer provides a meaningful description of the reality.

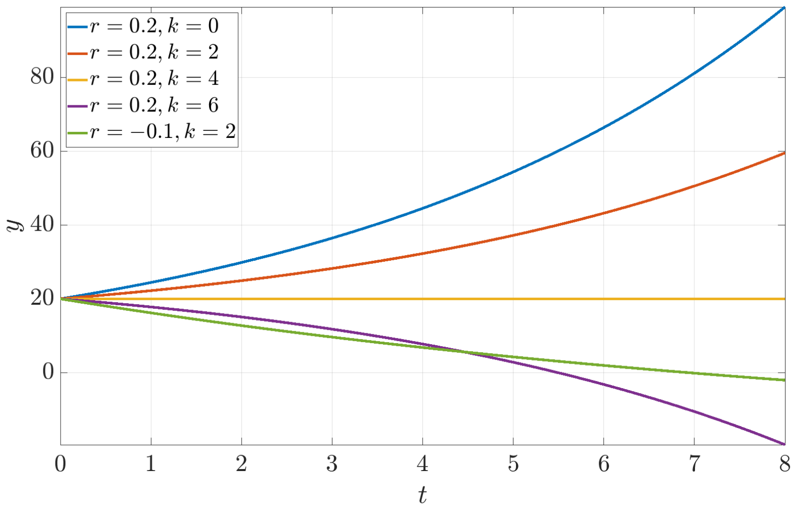
For $r < 0$ we write $r = -|r|$, from this we get

$$y(t) = \underbrace{\left(y_0 + \frac{k}{|r|}\right)\mathrm{e}^{-|r|t}}_{\to 0} - \frac{k}{|r|}.$$

We see that y decreases for all $k > 0$ and for t sufficiently large t it assumes negative values.

If $ry_0 = k$, the population remains constant at the value k/r.

(d) Sketch the solution for $y_0 = 20$ with $r = 0.2$ and $k = 0$, $k = 2$, $k = 4$, $k = 6$, as well as with $r = -0.1$ and $k = 2$.



(e) Suppose now $r < 0$. Can a growing population be obtained by adding a constant number $c > 0$ to the population in each time unit? Can it grow indefinitely in this case?

Solution. For $c > 0$, we consider the initial value problem

$$\begin{cases} y'(t) = ry(t) + c & \text{for } t > 0, \\ y(0) = y_0 & \text{for } t = 0. \end{cases}$$

This has the solution

$$y(t) = \left(y_0 + \frac{c}{r}\right) \mathrm{e}^{rt} - \frac{c}{r}.$$

Being $r < 0$, we write again $r = -|r|$ and with this

$$y(t) = \left(y_0 - \frac{c}{|r|}\right) \mathrm{e}^{-|r|t} + \frac{c}{|r|}.$$

Since

$$y'(t) = -|r| \left(y_0 - \frac{c}{|r|}\right) \mathrm{e}^{-|r|t},$$

we see that y grows in case $\left(y_0 - \frac{c}{|r|}\right) < 0$, that is if $c > |r|y_0$. This means that the constant increase c must be larger than the amount of the decrease in the original population. Since

$$\left(y_0 - \frac{c}{|r|}\right) \mathrm{e}^{-|r|t} \xrightarrow{t\to\infty} 0,$$

however, y does not grow without limit in this case either, but remains limited by $c/|r|$.