Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Homework 1

We consider a population of fish in the sea, which we denote by $y(t)$ at time $t \geq 0$. We assume as first (at least for large times unrealistic) step that there is an unlimited amount of food and space available to the fish, and that they are not influenced by any external factors (predators, fishing, etc.). Then the number of fish only varies due to natural births and deaths.

We assume that there is a constant birth rate $m \in [0,1]$ and a constant death-rate $n \in [0,1]$, respectively. For a short time step $\Delta t > 0$ we can describe the evolution of the population by

$$y(t+\Delta t) = y(t) + \Delta t\,(m \cdot y(t) - n \cdot y(t))\,. \tag{1}$$

(a) Deduce from this a differential equation. Solve the corresponding initial value problem with initial value $y(0) = y_0 > 0$. Show that with the *reproduction rate*

$$r := m - n, \qquad r \in [-1,1],$$

for the solution y of the initial value problem it holds:

$$\lim_{t\to\infty} y(t) \;= \begin{cases} \infty & \text{for } r > 0, \\ y_0 & \text{for } r = 0, \\ 0 & \text{for } r < 0. \end{cases}$$

In the following we suppose that an any time unit a constant number $k > 0$ of fish is caught. The corresponding initial value problem reads

$$\begin{cases} y'(t) = ry(t) - k & \text{for } t > 0, \\ y(0) = y_0 & \text{for } t = 0. \end{cases}$$

(b) Solve this initial value problem.

(c) Determine, according to k, r and y_0, if the population increases or decreases. Can it happen that the population remains constant?

Could negative values of y occur? How could these be possibly interpreted?

(d) Sketch the solution for $y_0 = 20$ with $r = 0.2$ and $k = 0,\ k = 2,\ k = 4,\ k = 6$, as well as with $r = -0.1$ and $k = 2$.

(e) Suppose now $r < 0$. Can a growing population be obtained by adding a constant number $c > 0$ to the population in each time unit? Can it grow indefinitely in this case?

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