Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Work sheet 6 - Solutions

Problem 1:

(a) Determine the equilibria of the following systems of differential equations. Check whether the equilibria are unstable, stable, or asymptotically stable.

(i) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 7 \\ -9 \end{pmatrix}$, (ii) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

(iii) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} -1 & -2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 6 \\ 6 \end{pmatrix}$, (iv) $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

(b) For which $\alpha \in \mathbb{R}$ is $(0,0,0)^\top \in \mathbb{R}^3$ a stable equilibrium of $u' = Au$? Here, A is respectively given by

(i) $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & \alpha & -1 \\ 0 & 1 & \alpha \end{pmatrix}$, (ii) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \alpha & -1 \\ 0 & 1 & \alpha \end{pmatrix}$, (iii) $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix}$.

Solution.

(a) (i) Equilibrium:

$$\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} + \begin{pmatrix} 7 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

Eigenvalues of the system matrix:

$$\det \begin{pmatrix} 1-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} = (1-\lambda)(-3-\lambda) - 5 = (\lambda+4)(\lambda-2) = 0$$
$$\Rightarrow \quad \lambda_1 = -4, \quad \lambda_2 = 2.$$

One eigenvalue has a positive real part. The equilibrium is *unstable*.

(ii) Equilibrium:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Eigenvalues of the system matrix:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = 0$$
$$\Rightarrow \quad \lambda_1 = 1 + \mathrm{i}, \quad \lambda_2 = 1 - \mathrm{i}.$$

One eigenvalue has a positive real part. The equilibrium is *unstable*.

(iii) Equilibrium:

$$\begin{pmatrix} -1 & -2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} - \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Eigenvalues of the system matrix:

$$\det \begin{pmatrix} -1-\lambda & -2 \\ 5 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda) + 10 = \lambda^2 + 9 = 0$$
$$\Rightarrow \quad \lambda_1 = 3\mathrm{i}, \quad \lambda_2 = -3\mathrm{i}.$$

No eigenvalue has positive real part. The eigenvalues with real part zero are simple. The equilibrium is *stable*.

(iv) Equilibrium:

$$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Eigenvalues of the system matrix:

$$\det \begin{pmatrix} -1-\lambda & 0 \\ -1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2$$
$$\Rightarrow \quad \lambda_1 = \lambda_2 = -1.$$

All eigenvalues has negative real part. The equilibrium is *asymptotically stable*.

(b) (i) Eigenvalues of the system matrix:

$$\det \begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & \alpha-\lambda & -1 \\ 0 & 1 & \alpha-\lambda \end{pmatrix} = (-2-\lambda)((\alpha-\lambda)^2+1) = 0$$
$$\Rightarrow \lambda_1 = -2,\ \lambda_2 = \alpha + \mathrm{i},\ \lambda_2 = \alpha - \mathrm{i}.$$

Zero is a stable equilibrium if $\alpha \leq 0$ and even asymptotically stable for all $\alpha < 0$.

(ii) We immediately see that $\lambda = 2$ is an eigenvalues. Therefore, the equilibrium is unstable, independently of α.

(iii) Eigenvalues of the system matrix:

$$\det \begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & \alpha-\lambda & 1 \\ 0 & 1 & \alpha-\lambda \end{pmatrix} = (-2-\lambda)((\alpha-\lambda)^2-1) = 0$$
$$\Rightarrow \lambda_1 = -2,\ \lambda_2 = \alpha + 1,\ \lambda_2 = \alpha - 1.$$

Zero is a stable equilibrium if $\alpha \leq -1$ and even asymptotically stable for all $\alpha < -1$.

Problem 2:

(a) (*Problem from an old exam, 4 points)* Consider the initial value problem

$$u''(t) + 4u'(t) + 3u(t) = 2\cos(t) + t^2 \mathrm{e}^{-2t} \qquad \text{for} \;\; t > 0,$$

with

$$u(0) = 0, \qquad u'(0) = 5.$$

What algebraic equation do you get when you apply the Laplace transform to this problem?

(b) Let $F(s) = \dfrac{1}{s(s+1)^2}$ be the image of the function $f : (0, \infty) \to \mathbb{R},\ t \mapsto f(t)$ under the Laplace transform. Determine f .

Solution.

(a) Let U be the image of u under the Laplace transform. Then we have:

$$u \circ\!\!-\!\!\bullet\, U, \qquad u' \circ\!\!-\!\!\bullet\, sU - u(0) \;=\; sU$$
$$u'' \circ\!\!-\!\!\bullet\, s^2 U - su(0) - u'(0) \;=\; s^2 U - 5.$$

Moreover,

$$\cos(t) \circ\!\!-\!\!\bullet\, \frac{s}{s^2+1}, \qquad t^2 \circ\!\!-\!\!\bullet\, \frac{2!}{s^{2+1}}, \qquad t^2 \mathrm{e}^{-2t} \circ\!\!-\!\!\bullet\, \frac{2}{(s+2)^3}.$$

The initial vale problem is transformed into the algebraic equation

$$(s^2 + 4s + 3)U - 5 = \frac{2s}{s^2+1} \;+\; \frac{2}{(s+2)^3}.$$

(b) We take the following ansatz for a partial fraction decomposition:

$$\frac{as+b}{s^2+2s+1} + \frac{c}{s} \;=\; \frac{1}{s(s+1)^2}.$$

we get

$$cs^2 + 2cs + c + as^2 + bs = 1 \quad \Rightarrow \quad c = -a, \;\; -2c = b,\; c = 1,$$

and thus

$$\frac{-s-2}{(s+1)^2} + \frac{1}{s} \;=\; -\frac{1}{(s+1)^2} \;-\; \frac{s+1}{(s+1)^2} \;+\; \frac{1}{s}$$
$$= -\frac{1}{(s+1)^2} \;-\; \frac{1}{s+1} \;+\; \frac{1}{s} \,\bullet\!\!-\!\!\circ\, -t\mathrm{e}^{-t} - \mathrm{e}^{-t} + 1 \;=\; f(t).$$