Differential Equations I for Students of Engineering Sciences

Work sheet 5 - Solutions

Problem 1. Consider the matrix $A \in \mathbb{R}^{3\times 3}$,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 4 & -3 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Compute the eigenvalues of A. Determine the corresponding eigenvectors, and, if necessary, the generalized eigenvectors.

(b) Determine a fundamental system for $u' = Au$.

Solution.

(a) Eigenvalues:

$$\begin{aligned} p(\lambda) &= \det \begin{pmatrix} -\lambda & 0 & 1 \\ 4 & -3-\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda \det \begin{pmatrix} -3-\lambda & 0 \\ 1 & -\lambda \end{pmatrix} + \det \begin{pmatrix} 4 & -3-\lambda \\ 0 & 1 \end{pmatrix} \\ &= -\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2 \qquad \Rightarrow \qquad \lambda_1 = 1, \quad \lambda_{2,3} = -2 \end{aligned}$$

Eigenvector $v^{[1]}$ for $\lambda_1 = 1$:

$$\left(\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 4 & -4 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \Rightarrow v^{[1]} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Eigenvectors for $\lambda_{2,3} = -2$:

$$\left(\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{rrr|r} 2 & 0 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \Rightarrow v^{[2]} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}.$$

The corresponding eigenspace has dimension one and thus, the geometric multiplicity of the eigenvalue $\lambda_{2,3} = -2$ is smaller than its algebraic multiplicity. We need a generalized eigenvector of the second step , which we find through the ansatz

$$(A - \lambda_{2,3} I)v^{[3]} = v^{[2]}.$$

We have

$$\left(\begin{array}{rrr|r} 2 & 0 & 1 & -1 \\ 4 & -1 & 0 & -4 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad \to \quad \left(\begin{array}{rrr|r} 2 & 0 & 1 & -1 \\ 0 & -1 & -2 & -2 \\ 0 & 1 & 2 & 2 \end{array}\right) \quad \Rightarrow \quad v^{[3]} = \begin{pmatrix} -1/2 \\ 2 \\ 0 \end{pmatrix}.$$

(b) A fundamental system is:

$$u_1(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ u_2(t) = e^{-2t} \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix},\ u_3(t) = e^{-2t} \left\{ t \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 2 \\ 0 \end{pmatrix} \right\}.$$

Problem 2.

(a) We consider the inhomogeneous system

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \begin{pmatrix} 3 & 4 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 10t\mathrm{e}^{-3t} \end{pmatrix}.$$

Find a fundamental system for the homogeneous problem. Find a particular solution of the inhomogeneous problem by the method of variation of constants.

(b) We consider the inhomogeneous system

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}' = \begin{pmatrix} -6 & -4 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \mathrm{e}^{2t} \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

Determine a *real* fundamental system for the homogeneous problem.Find a particular solution of the inhomogeneous problem by using the ansatz
$u_p(t) = \mathrm{e}^{2t}(a,b)^\top$, with suitable $a, b \in \mathbb{R}$.

Solution.

(a) Consider the homogeneous problem first.

Eigenvalues:

$$p(\lambda) = \det \begin{pmatrix} 3-\lambda & 4 \\ 6 & 1-\lambda \end{pmatrix} = (3-\lambda)(1-\lambda) - 24 = \lambda^2 - 4\lambda - 21$$

$$\Rightarrow \qquad \lambda_{1,2} = 2 \pm \sqrt{4+21} = 2 \pm 5 \qquad \Rightarrow \qquad \lambda_1 = 7, \quad \lambda_2 = -3.$$

Eigenvector for $\lambda_1 = 7$:

$$\left(\begin{array}{rr|r} -4 & 4 & 0 \\ 6 & -6 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{rr|r} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad v^{[1]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = -3$:

$$\left(\begin{array}{cc|c} 6 & 4 & 0 \\ 6 & 4 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad v^{[2]} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

A fundamental system is $\{e^{7t}(1,1)^\top,\ e^{-3t}(-2,3)^\top\}$ and the general solution is

$$u_h(t) = c_1 e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R},$$

which we can express through the fundamental matrix:

$$u_h(t) = \begin{pmatrix} e^{7t} & -2e^{-3t} \\ e^{7t} & 3e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

For the variation of constants the ansatz

$$u_p(t) = K_1(t) e^{\lambda_1 t} v^{[1]} + K_2(t) e^{\lambda_2 t} v^{[2]}$$

leads to the conditions

$$\begin{aligned} e^{7t} K_1'(t) - 2e^{-3t} K_2'(t) &= 0, \\ e^{7t} K_1'(t) + 3e^{-3t} K_2'(t) &= 10t e^{-3t}. \end{aligned}$$

We subtract the first equation from the first to arrive at

$$5e^{-3t} K_2'(t) = 10t e^{-3t} \quad \Rightarrow \quad K_2'(t) = 2t,$$

so we can choose $K_2(t) = t^2$ as a primitive.
Plugging this into the first equation yields

$$e^{7t} K_1'(t) - 2e^{-3t} K_2'(t) = e^{7t} K_1'(t) - 4e^{-3t} t = 0 \quad \Rightarrow \quad K_1'(t) = 4t e^{-10t}$$

and after a short calculation (e.g. using integration by parts) we get

$$K_1(t) = -\frac{1}{25}(10t+1) e^{-10t}.$$

Finally, this gives us

$$\begin{aligned} u_p(t) &= -\frac{1}{25}(10t+1) e^{-10t} \cdot e^{7t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t^2 \cdot e^{-3t} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ &= \frac{1}{25 e^{3t}} \begin{pmatrix} -50t^2 - 10t - 1 \\ 75t^2 - 10t - 1 \end{pmatrix}. \end{aligned}$$

(b) Consider the homogeneous problem first.

Eigenvalues:

$$p(\lambda) = \det \begin{pmatrix} -6-\lambda & -4 \\ 5 & 2-\lambda \end{pmatrix} = (-6-\lambda)(2-\lambda) + 20 = \lambda^2 + 4\lambda + 8$$

$$\Rightarrow \qquad \lambda_{1,2} = -2 \pm \sqrt{4-8} = 2 \pm 2i \qquad \Rightarrow \qquad \lambda_1 = -2 + 2i, \quad \lambda_2 = -2 - 2i.$$

Eigenvector for $\lambda_1 = -2+2\mathrm{i}$:

$$\begin{pmatrix} -4-2\mathrm{i} & -4 \\ 5 & 4-2\mathrm{i} \end{pmatrix} \begin{pmatrix} v_1^{[1]} \\ v_2^{[1]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

It follows

$$(-4-2\mathrm{i})v_1^{[1]} - 4v_2^{[1]} = 0 \quad \Rightarrow \quad v_2^{[1]} = (-1-\frac{1}{2}\mathrm{i})v_1^{[1]},$$

and

$$5v_1^{[1]} + (4-2\mathrm{i})v_2^{[1]} = 0,$$

which does not provide any further information. We choose

$$v^{[1]} = \begin{pmatrix} 2 \\ -2-\mathrm{i} \end{pmatrix}, \qquad v^{[2]} = \bar{v}^{[1]} = \begin{pmatrix} 2 \\ -2+\mathrm{i} \end{pmatrix}.$$

A complex fundamental system is then given by

$$\left\{ \mathrm{e}^{(-2-2\mathrm{i})t} \begin{pmatrix} 2 \\ -2-\mathrm{i} \end{pmatrix}, \qquad \mathrm{e}^{(-2+2\mathrm{i})t} \begin{pmatrix} 2 \\ -2+\mathrm{i} \end{pmatrix} \right\}.$$

We get a real fundamental system by taking the real part and the imaginary part, respectively, of the first fundamental solution:

$$\begin{aligned} \mathrm{e}^{(-2-2\mathrm{i})t} \begin{pmatrix} 2 \\ -2-\mathrm{i} \end{pmatrix} &= \mathrm{e}^{-2t}(\cos(2t) - \mathrm{i}\sin(2t)) \begin{pmatrix} 2 \\ -2-\mathrm{i} \end{pmatrix} \\ &= e^{-2t} \begin{pmatrix} 2\cos(2t) - 2\mathrm{i}\sin(2t) \\ -2\cos(2t) + 2\mathrm{i}\sin(2t) - \mathrm{i}\cos(2t) - \sin(2t) \end{pmatrix} \\ &= \underbrace{e^{-2t} \begin{pmatrix} 2\cos(2t) \\ -2\cos(2t) - \sin(2t) \end{pmatrix}}_{=:w_1(t)} + \mathrm{i} \cdot \underbrace{e^{-2t} \begin{pmatrix} 2\sin(2t) \\ 2\sin(2t) - \cos(2t) \end{pmatrix}}_{=:w_2(t)}, \end{aligned}$$

and $\{w_1, w_2\}$ is a real fundamental system.

The ansatz $u_p(t) = \mathrm{e}^{2t}(a,b)^\top$, leads to

$$\begin{aligned} u_p' = 2\mathrm{e}^{2t} \begin{pmatrix} a \\ b \end{pmatrix} &\overset{!}{=} \mathrm{e}^{2t} \begin{pmatrix} -6 & -4 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \mathrm{e}^{2t} \begin{pmatrix} 2 \\ -5 \end{pmatrix} \\ \Rightarrow \quad 2 \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 6 & 4 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 2 \\ -5 \end{pmatrix}. \end{aligned}$$

Therefore, $8a + 4b = 2$ and $-5a = -5$, which gives $a = 1$ and $b = -\frac{3}{2}$.