Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Differential Equations I for Students of Engineering Sciences

Work sheet 4 - Solutions

Problem 1: Find the general solutions of the following differential equations:

(a) $u'''(t) - u'(t) = 0\,;$

(b) $u'''(t) - 5u''(t) + 8u'(t) - 4u(t) = 0\,;$

(c) $u''(t) - 2u'(t) + 5u(t) = 0\,.$

Solution:

(a) The characteristic polynomial is

$$p(\lambda) = \lambda^3 - \lambda = \lambda(\lambda^2 - 1) = \lambda(\lambda + 1)(\lambda - 1),$$

with zeros $\lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = +1$. The functions $u_k(t) = \mathrm{e}^{\lambda_k t}$ with $1 \le k \le 3$, form a fundamental system. Therefore, the general solution is

$$u(t) = c_1 + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{t}, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

(b) The characteristic polynomial is

$$p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

The coefficients sum to zero, so $\lambda = 1$ is a root. By polynomial division we find

$$p(\lambda) = (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2.$$

We see that $\lambda = 2$ is a zero with multiplicity 2. Therefore, the functions

$$u_1(t) = \mathrm{e}^{t}, \quad u_2(t) = \mathrm{e}^{2t}, \quad u_3(t) = t\mathrm{e}^{2t}$$

form a fundamental system and the general solution is

$$u(t) = c_1\mathrm{e}^{t} + c_2\mathrm{e}^{2t} + c_3 t\mathrm{e}^{2t}, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

(c) The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 2\lambda + 5,$$

with zeros

$$\lambda_{1,2} \;=\; 1 \pm \sqrt{1-5} \;=\; 1 \pm 2\mathrm{i}.$$

We get a real fundamental system from $u_1(t) = \mathrm{Re}(\mathrm{e}^{\lambda_1 t})$ and $u_2(t) = \mathrm{I}m(\mathrm{e}^{\lambda_1 t})$. By the Euler identity:

$$\mathrm{e}^{(1+2\mathrm{i})t} = \mathrm{e}^t \mathrm{e}^{2\mathrm{i}t} = \mathrm{e}^t(\cos(2t) + \mathrm{i}\sin(2t)) \quad\Rightarrow\quad u_1(t) = \mathrm{e}^t\cos(2t), \quad u_2(t) = \mathrm{e}^t\sin(2t).$$

The general solution is

$$u(t) = \mathrm{e}^t \cdot (c_1\cos(2t) + c_2\sin(2t)), \qquad c_1, c_2 \in \mathbb{R}.$$

Problem 2 (*problem from an old exam, 4 points)*: Consider the third order differential equation

$$u'''(t) + a_2u''(t) + a_1u'(t) + a_0u(t) = 0 \tag{$*$}$$

with real coefficients $a_0, a_1, a_2 \in \mathbb{R}$. Check whether the following sets of functions can be fundamental systems for the solution space of this equation (with suitable coefficients $a_0, a_1, a_2 \in \mathbb{R}$). Give an explanation for your answers.

(a) $M_1 := \{u_1(t) = -t,\ u_2(t) = 1,\ u_3(t) = 2t\}$.

(b) $M_2 := \{u_1(t) = \mathrm{e}^{-t},\ u_2(t) = \mathrm{e}^t,\ u_3(t) = \mathrm{e}^{2t},\ u_4(t) = \mathrm{e}^{3t}\}$.

(c) $M_3 := \{u_1(t) = \mathrm{e}^{-t},\ u_2(t) = e^{\mathrm{i}t},\ u_3(t) = e^{2\mathrm{i}t}\}$.

(d) $M_4 := \{u_1(t) = 1,\ u_2(t) = e^{-2\mathrm{i}t},\ u_3(t) = e^{2\mathrm{i}t}\}$

Solution:

(a) The set M_1 cannot be a fundamental system, because it is not linearly independent. For example: $u_3(t) + 0\cdot u_2(t) + 2u_1(t) = 0$. The space spanned by M_1 has dimension two.

(b) The solution space has dimension three, but M_2 spans a space of dimension four. Therefore, M_2 cannot be a fundamental system.

(c) If a linear ODE with constant real coefficients has complex solutions, then the complex solutions appear in pairs of complex conjugates. Therefore, M_3 cannot be a fundamental system for $(*)$.

(d) The set M_4 is a fundamental system for $(*)$ with suitable coefficients. It spans a three-dimensional space, and the complex basis functions are a pair of complex conjugates.

Not part of this problem: In fact, the characteristic polynomial that would lead to M_4 is $p(\lambda) = \lambda(\lambda^2 + 4)$, which corresponds to the equation $u'''(t) + 4u'(t) = 0$.

Problem 3: Solve the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \qquad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution. We compute the eigenvalues and eigenvectors of the system matrix:

$$0 \;=\; \det\begin{pmatrix} -2-\lambda & 1 \\ 3 & -4-\lambda \end{pmatrix} \;=\; (2+\lambda)(4+\lambda)-3 \;=\; \lambda^2+6\lambda+5$$
$$\Rightarrow \quad \lambda_1 = -1, \qquad \lambda_2 = -5.$$

Eigenvectors:

for $\lambda_1 = -1$: $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ 3 & -3 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $\lambda_2 = -5$: $\left(\begin{array}{cc|c} 3 & 1 & 0 \\ 3 & 1 & 0 \end{array}\right) \quad \to \quad \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

The general solution of the system with $c_1, c_2 \in \mathbb{R}$ is:

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \;=\; c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

From the initial data we get

$$c_1 + c_2 = 3, \quad c_1 - 3c_2 = 2 \qquad \Rightarrow \qquad c_1 = \frac{11}{4}, \quad c_2 = \frac{1}{4}.$$

Finally, the solution of the initial value problem is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \;=\; \frac{11}{4}e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \;+\; \frac{1}{4}e^{-5t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$