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Differential Equations I for Students of Engineering Sciences

Work Sheet 3 - Solutions

Problem 1:

(a) Consider the equation

$$y''(t) - 8y'(t) + 15y(t) = 0, \qquad t > 0.$$

Find $\lambda_1, \lambda_2 \in \mathbb{R}$, such that $y_1(t) := c_1 \mathrm{e}^{\lambda_1 t}$ and $y_2(t) := c_2 \mathrm{e}^{\lambda_2 t}$, with $c_1, c_2 \in \mathbb{R}$, are solutions of this equation. Is $y_1 + y_2$ also a solution?

(b) Now let the following *Euler differential equation* be given:

$$t^2 u''(t) - 7tu'(t) + 15u(t) = 0, \qquad t > 0,$$

Solve this equation by using a suitable change of variables to transform it to the equation from part (a).

Solution:

(a) Let y be a solution of the form $y(t) = c\mathrm{e}^{\lambda t}$ with $c, \lambda \in \mathbb{R}$. Then it holds

$$\lambda^2 c\mathrm{e}^{\lambda t} - 8\lambda c\mathrm{e}^{\lambda t} + 15c\mathrm{e}^{\lambda t} \;=\; (\lambda^2 - 8\lambda + 15)c\mathrm{e}^{\lambda t} \;=\; 0.$$

We get one solution for $c = 0$. For $c \neq 0$ we have $c\mathrm{e}^{\lambda t} \neq 0$ for all $t > 0$ and therefore

$$\lambda^2 - 8\lambda + 15 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 4 \pm \sqrt{16 - 15} \quad \Rightarrow \quad \lambda_1 = 3,\ \lambda_2 = 5.$$

Thus, $y_1(t) = c_1 \mathrm{e}^{3t}$ and $y_2(t) = c_2 \mathrm{e}^{5t}$ with $c_1, c_2 \in \mathbb{R}$ are solutions. Since the equation is linear, also $y_1 + y_2$ is a solution.

(b) Define $\mathrm{e}^s := t$ and $y(s) := u(\mathrm{e}^s)$. Then it holds

$$\begin{aligned}
&y'(s) = \mathrm{e}^s u'(\mathrm{e}^s) \quad \Rightarrow \quad u'(t) = u'(\mathrm{e}^s) = \mathrm{e}^{-s} y'(s),\\
&y''(s) = \mathrm{e}^s u'(\mathrm{e}^s) + \mathrm{e}^{2s} u''(\mathrm{e}^s) \quad \Rightarrow \quad u''(t) = u''(\mathrm{e}^s) = \mathrm{e}^{-2s}(y''(s) - y'(s)).
\end{aligned}$$

Plugging this into the differential equation yields

$$\begin{aligned}
0 \;=\; t^2 u''(t) - 7tu'(t) + 15u(t) \;&=\; t^2 \cdot t^{-2}(y''(s) - y'(s)) - 7t \cdot t^{-1} y'(s) + 15y(s)\\
&=\; y''(s) - 8y'(s) + 15y(s).
\end{aligned}$$

In part (a) we have seen that this equation for y has the solution

$$y(s) = c_1 e^{3s} + c_2 e^{5s}, \qquad c_1, c_2 \in \mathbb{R}.$$

Using the back-transformation $s = \ln(t)$ we finally arrive at´

$$u(t) = c_1 t^3 + c_2 t^5.$$

Problem 2: (*problem from an old exam, 5 points*)

(a) Check whether the following ordinary differential equations are exact.

  (i) $y(t)^2 + (t^2 y(t) - 1)y'(t) = 0\,;$

  (ii) $2ty(t)^2 + (2y(t) + 2t^2 y(t))y'(t) = 0\,.$

(b) Determine a corresponding scalar potential and the general solution for the exact equation from part (a).

Solution.

(a) (i) The equation has the form $f(t,y) + g(t,y)y'(t) = 0$ with $f(t,y) = y^2$ and $g(t,y) = t^2 y - 1$. It holds

$$f_y(t,y) \;=\; 2y \;\neq\; 2ty \;=\; g_t(t,y).$$

Therefore, the equation is *not* exact.

  (ii) The equation has the form $f(t,y) + g(t,y)y'(t) = 0$ mit $f(t,y) = 2ty^2$ und $g(t,y) = 2(t^2+1)y$. It holds

$$f_y(t,y) \;=\; 4ty \;=\; g_t(t,y),$$

Therefore, the equation is exact.

(b) We compute a scalar potential for the equation from part (a).(ii):

$$\begin{aligned} \Psi_t(t,y) = f(t,y) = 2ty^2 \quad &\Rightarrow \quad \Psi(t,y) = t^2 y^2 + D(y), \\ \Psi_y(t,y) = g(t,y) = 2(t^2+1)y \quad &\Rightarrow \quad \Psi(t,y) = (t^2+1)y^2 + K(t). \end{aligned}$$

We choose $K(t) = 0$ and $D(y) = y^2$. With that we have $\Psi(t,y) = (t^2+1)y^2$.

The solutions of the ODE are given by the contour lines $\Psi(t,y) = C$, i.e.

$$C = (t^2+1)y^2 \quad \Rightarrow \quad y(t) = \pm\sqrt{\frac{C}{t^2+1}}, \quad C > 0.$$

Problem 3: Show that the differential equation

$$(t^2 - 1)y + (t^3 + t)y' = 0, \qquad t > 0,$$

admits an integrating factor h that depends only on t (i.e. $h = h(t)$) and determine the solutions of the equation.

Solution: With $f(t,y) = (t^2-1)y,\ g(t,y) = t^3 + t$ we have

$$f_y = t^2 - 1 \ \neq\ 3t^2 + 1 = g_t,$$

and we see that the equation is not exact.

In order for $h(t)$ to be an integrating factor, we need

$$\frac{\partial}{\partial y}\left(h(t)\cdot f(t,y)\right) = \frac{\partial}{\partial t}\left(h(t)\cdot g(t,y)\right).$$

It holds

$$\frac{\partial}{\partial y}\left(h(t)\cdot f(t,y)\right) = h(t)\cdot f_y(t,y) = h(t)\cdot(t^2-1)$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(h(t)\cdot g(t,y)\right) &= h'(t)\cdot g(t,y) + h(t)\cdot g_t(t,y)\\ &= h'(t)\cdot(t^3+t) + h(t)\cdot(3t^2+1).\end{aligned}$$

Combing both we find:

$$\begin{aligned}&h(t)\cdot(t^2-1) = h'(t)\cdot(t^3+t) + h(t)\cdot(3t^2+1)\\ &\qquad\Rightarrow\quad h'(t)\cdot(t^3+t) = -2(t^2+1)h(t) \quad\Rightarrow\quad h'(t) = -\frac{2}{t}h(t).\end{aligned}$$

By separation of variables we compute $h(t) = \frac{1}{t^2}$ as one solution for h. With that the original ODE becomes

$$0 \ =\ h(t)\cdot(t^2-1)y + h(t)\cdot(t^3+t)y' \ =\ \left(1-\frac{1}{t^2}\right)y + \left(t+\frac{1}{t}\right)y'.$$

From that we immediately get

$$\int\left(1-\frac{1}{t^2}\right)y\ \mathrm{d}t \ =\ \left(t+\frac{1}{t}\right)y \ =\ \int\left(t+\frac{1}{t}\right)\mathrm{d}y,$$

and thus $\Psi(t,y) = (t+1/t)y$ is a corresponding scalar potential. The solutions are then given by

$$\left(t+\frac{1}{t}\right)y \ =\ C \qquad\Rightarrow\qquad y(t) \ =\ \frac{C}{t+\frac{1}{t}} \ =\ \frac{Ct}{t^2+1}.$$