**Fachbereich Mathematik der Universität Hamburg** **WiSe 2024/25**
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# Differential Equations I for Students of Engineering Sciences

## Exercise 2 - Solutions

**Problem 1:** Solve the following initial value problem:

$$y'(x) - 2y(x) = 1 + 4\mathrm{e}^{-2x}, \qquad y(0) = 1.$$

**Solution:** In standard form the differential equation reads

$$y'(x) = 2y(x) + 1 + 4\mathrm{e}^{-2x}.$$

So we have a linear, inhomogeneous first order equation with

$$a(x) = 2, \qquad b(x) = 1 + 4\mathrm{e}^{-2x}.$$

We choose $A(x) = 2x$ as an antiderivative of $a$ and get from the solution formula that

$$\begin{aligned}
y(x) &= \mathrm{e}^{A(x)} \left[ \int_{x_0}^{x} \mathrm{e}^{-A(s)} b(s)\ \mathrm{d}s + y_0 \mathrm{e}^{-A(x_0)} \right] = \mathrm{e}^{2x} \left[ \int_0^x \mathrm{e}^{-2s} \left(1 + 4\mathrm{e}^{-2s}\right)\ \mathrm{d}s + 1 \right] \\
&= \mathrm{e}^{2x} \int_0^x \mathrm{e}^{-2s}\ \mathrm{d}s + 4\mathrm{e}^{2x} \int_0^x \mathrm{e}^{-4s}\ \mathrm{d}s + \mathrm{e}^{2x} \\
&= \mathrm{e}^{2x} \cdot \left( -\frac{1}{2}\mathrm{e}^{-2s} \right)\Big|_0^x + 4\mathrm{e}^{2x} \cdot \left( -\frac{1}{4}\mathrm{e}^{-4s} \right)\Big|_0^x + \mathrm{e}^{2x} \\
&= -\frac{1}{2}\mathrm{e}^{2x}\left(\mathrm{e}^{-2x} - 1\right) - \mathrm{e}^{2x}\left(\mathrm{e}^{-4x} - 1\right) + \mathrm{e}^{2x} = -\frac{1}{2} + \frac{1}{2}\mathrm{e}^{2x} - \mathrm{e}^{-2x} + \mathrm{e}^{2x} + \mathrm{e}^{2x} \\
&= -\mathrm{e}^{-2x} + \frac{5}{2}\mathrm{e}^{2x} - \frac{1}{2}.
\end{aligned}$$

**Problem 2:** Use the method of separation of variables to solve the following equations:

$(a)\ \ y' = x^2 y, \qquad (b)\ \ y' = xy^2, \qquad (c)\ \ y' = (1-\sin(x))y, \qquad (d)\ \ y' = \dfrac{x\cos^2(y)}{1+x^2}.$

**Solution:**

(a) One solution is $y = 0$. For $y \neq 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y \quad \Rightarrow \quad \int \frac{1}{y}\,\mathrm{d}y = \int x^2 \mathrm{d}x \quad \Rightarrow \quad \ln(|y|) = \frac{1}{3}x^3 + \tilde{c}, \qquad \tilde{c} \in \mathbb{R},$$
$$\Rightarrow \quad |y| = \mathrm{e}^{\tilde{c}} \cdot \mathrm{e}^{x^3/3} \quad \Rightarrow \quad y(x) = c\mathrm{e}^{x^3/3}, \qquad c \in \mathbb{R}.$$

(b) One solution is $y = 0$. For $y \neq 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 \quad \Rightarrow \quad \int \frac{1}{y^2}\,\mathrm{d}y = \int x\,\mathrm{d}x \quad \Rightarrow \quad -\frac{1}{y} = \frac{1}{2}x^2 + c$$
$$\Rightarrow \quad y(x) = \frac{1}{c - \frac{x^2}{2}}, \qquad c \in \mathbb{R}.$$

(c) One solution is $y = 0$. For $y \neq 0$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (1-\sin(x))y \quad \Rightarrow \quad \int \frac{1}{y}\,\mathrm{d}y = \int (1-\sin(x))\,\mathrm{d}x$$
$$\Rightarrow \quad \ln(|y|) = x + \cos(x) + \tilde{c}, \qquad \tilde{c} \in \mathbb{R}$$
$$\Rightarrow \quad |y| = \mathrm{e}^{\tilde{c}} \cdot \mathrm{e}^{x+\cos(x)} \quad \Rightarrow \quad y(x) = c\mathrm{e}^{x+\cos(x)}, \qquad c \in \mathbb{R}.$$

(d) For $k \in \mathbb{Z}$, the constant function $y = \frac{\pi}{2} + k\pi$ satisfies $\cos^2(y) = 0$ and $y' = 0$. Otherwise we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\cos^2(y)}{1+x^2} \quad \Rightarrow \quad \int \frac{1}{\cos^2(y)}\,\mathrm{d}y = \int \frac{x}{1+x^2}\,\mathrm{d}x$$
$$\Rightarrow \quad \tan(y) = \frac{1}{2}\ln\left(1+x^2\right) + c$$
$$\Rightarrow \quad y(x) = \arctan\left(\frac{1}{2}\ln\left(1+x^2\right) + c\right), \qquad c \in \mathbb{R}.$$

With this we have $y(x) \in (-\pi/2, \pi/2)$ for all $x \in \mathbb{R}$ and therefore $\cos^2(y) \neq 0$.

The function $\cos^2(y)$ is $\pi$-periodic, i.e. $\cos^2(y+k\pi) = \cos^2(y)$ for all $k \in \mathbb{Z}$. Moreover, it holds $\frac{\mathrm{d}}{\mathrm{d}x}(y(x)+k\pi) = \frac{\mathrm{d}y(x)}{\mathrm{d}x}$. Therefore, for each solution $y$, also $y+k\pi$ with $k \in \mathbb{Z}$ is a solution.

**Problem 3:** Solve the following initial value problem for *Bernoulli* differential equation:

$$u' = \frac{1}{3}u + \frac{1}{3}u^4 \qquad \text{for} \quad t > 0, \qquad u(0) = 1.$$

Is the solution defined for all $t > 0$?

**Solution:** This is a Bernoulli equation with $a = b = \frac{1}{3}$ and $\alpha = 4$. By substituting $y(t) := u^{1-\alpha}(t) = u^{-3}(t)$ we find

$$y(t)' = (1-\alpha)\,[a(t)y(t) + b(t)] = -3\left[\frac{1}{3}y(t) + \frac{1}{3}\right] = -y(t) - 1,$$

and $y(0) = u^{-3}(0) = 1$.

Using the solution formula for linear first order equations we compute:

$$y(t) = \mathrm{e}^{-t}\left[\int_0^t \mathrm{e}^s \cdot (-1)\,\mathrm{d}s + 1\right] = \mathrm{e}^{-t}\left[-(\mathrm{e}^t - 1) + 1\right] = 2\mathrm{e}^{-t} - 1.$$

Now we have to transform back from $y$ to $u$:

$$y = u^{-3} \qquad \Leftrightarrow \qquad u = y^{-1/3} = \frac{1}{\sqrt[3]{y}},$$

so

$$u(t) = \frac{1}{\sqrt[3]{2\mathrm{e}^{-t} - 1}}.$$

The solution is only defined for $2\mathrm{e}^{-t} - 1 \neq 0$. That is, for $t < \ln(2)$.