Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. T. Schmidt, Dr. C. Goetz

Differential Equations I for Students of Engineering Sciences

Exercise class 1 - Solutions

Exercise 1:

(a) We consider the differential equation

$$u'(t) = u(t) - t \qquad \text{for } t \in \mathbb{R}.$$

Which of the following functions defined on $\mathbb{R}$ are solutions of such differential equation ?

(1) $u_1(t) := \mathrm{e}^t$, (2) $u_2(t) := t + 1$,
(3) $u_3(t) := \alpha \mathrm{e}^t + t + 1$, (4) $u_4(t) := \mathrm{e}^t + \beta(t+1)$,

where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants.

(b) Given the differential equation

$$y''(t) + 16y(t) = 0 \qquad \text{for } t \in \mathbb{R},$$

show that the functions

$$y_1(t) := \cos(4t), \qquad y_2(t) := \sin(4t), \qquad y_3(t) := \alpha\cos(4t) + \beta\sin(4t)$$

defined on $\mathbb{R}$ for arbitrary $\alpha, \beta \in \mathbb{R}$, are solutions.

Does the initial value problem $y'' + 16y = 0,\ y(0) = 0$ have a unique solution?

Solution:

(a) We check directly:
- (1) $u_1'(t) = e^t \ \neq \ e^t - t = u_1(t) - t$, thus u_1 is *not* a solution.
- (2) $u_2'(t) = 1 \ = \ (1+t) - t = u_2(t) - t$, thus u_2 is a solution.
- (3) $u_3'(t) = \alpha\mathrm{e}^t + 1 \ = \ u_3(t) - t$, thus u_3 is a solution.
- (4) $u_4'(t) = \mathrm{e}^t + \beta,\ u_4(t) - t = \mathrm{e}^t + (\beta - 1)t + \beta$, thus u_4 is a solution only for $\beta = 1$.

(b) It holds

$$\begin{aligned} y_1(t) &= \cos(4t), & y_1'(t) &= -4\sin(4t), & y_1''(t) &= -16\cos(4t),\\ y_2(t) &= \sin(4t), & y_2'(t) &= 4\cos(4t), & y_2''(t) &= -16\sin(4t), \end{aligned}$$

thus it is $y_1'' + 16y_1 = 0$ and $y_2'' + 16y_2 = 0$.

Because of the linearity of the derivative, we also obtain

$$y_3''(t) = -16\alpha\cos(4t) - 16\beta\sin(4t) = -16y_3(t).$$

The linear combination of two solutions is again a solution, since the equation is linear!

Choosing $\alpha = 0$, then the function $y(t) = \beta\sin(4t)$ satisfies for every $\beta \in \mathbb{R}$ both the differential equation and the condition $y(0) = 0$. This is due to the fact that this is a second-order equation. For equations of higher order, generally we need more conditions.

Exercise 2: We denote the humidity of a cloth in a drying machine at time $t \geq 0$ with $m(t)$, and let $m_0 := m(0) > 0$. It holds:

- During drying, the decrease in humidity is proportional to the humidity.
- After 15 minutes the cloth has still $50\,\%$ of its original humidity m_0.

Set an appropriate differential equation describing this process and determine its solution. How long does it take until the cloth has just the $2\,\%$ of its original humidity?

Solution: From the first assumption it follows that m can be described by

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -\lambda m, \qquad m(0) = m_0.$$

Since we assume that the humidity decreases, we choose $\lambda > 0$.

This problem has the solution

$$m(t) = m_0 \mathrm{e}^{-\lambda t}.$$

To determine λ we employ the second condition:

$$0.5m_0 = m(15) = m_0\mathrm{e}^{-15\lambda} \quad\Rightarrow\quad 0.5 = \mathrm{e}^{-15\lambda} \quad\Rightarrow\quad \lambda = -\frac{\ln(0.5)}{15} \approx 0.046.$$

Let t_* be the time at which the cloth has only $2\,\%$ of its original humidity. Then it holds

$$\begin{aligned} m(t_*) &= 0.02m_0 = m_0 \exp\left(\frac{\ln(0.5)}{15}t_*\right) \\ \Rightarrow \quad \ln(0.02) &= \frac{\ln(0.5)}{15}t_* \quad\Rightarrow\quad t_* = 15\cdot\frac{\ln(0.02)}{\ln(0.5)} \approx 84.66. \end{aligned}$$

Exercise 3: We consider the function $f : [0,\infty) \longrightarrow \mathbb{R}$, defined by

$$f(u) := (1-u)u.$$

(a) Show that the function φ_c defined by

$$\varphi_c(t) := \frac{e^t}{e^t + c},$$

with $c \in \mathbb{R}$ is a solution of the differential equation $\varphi_c' = f(\varphi_c)$ for $t \geq 0$, provided $e^t + c > 0$.

(b) Assume now that an initial value $u(0) = u_0 > 0$ is given. How should c from Part (a) be chosen, such that $u(t) = \varphi_c(t)$ solves the initial value problem $u' = f(u)$, $u(0) = u_0$? What are the solutions for $u_0 = 0$ and for $u_0 = 1$?

(c) Are the solutions in (b) with $u_0 > 0$ defined for every $t > 0$? Meaning, is the condition $e^t + c > 0$ satisfied for all $t > 0$?

(d) Determine the solutions of the initial value problem $u' = f(u)$, $u(0) = u_0$ for $u_0 = \frac{1}{2}$ and for $u_0 = \frac{3}{2}$. Sketch these solutions.

How do the solutions behave for $t \to \infty$?

Solution:

(a) It holds

$$\varphi_c'(t) = \frac{e^t(e^t + c) - e^t \cdot e^t}{(e^t + c)^2} = \frac{ce^t}{(e^t + c)^2}.$$

Moreover we have

$$f(\varphi_c(t)) = \varphi_c(t) - \varphi_c^2(t) = \frac{e^t(e^t + c)}{(e^t + c)^2} - \left(\frac{e^t}{e^t + c}\right)^2 = \frac{ce^t}{(e^t + c)^2}.$$

Thus it is $\varphi_c'(t) = f(\varphi_c(t))$.

(b) Let $u_0 > 0$. In order for $\varphi_c(t)$ with a $c \in \mathbb{R}$ to solve the initial value problem, it must be

$$u_0 = \varphi_c(0) = \frac{e^0}{e^0 + c} = \frac{1}{1 + c},$$

which is fulfilled for

$$c = \frac{1 - u_0}{u_0}.$$

For $u_0 = 1$ we get $c = 0$ and therefore the constant solution $u = 1$.

For $u_0 = 0$ we cannot get a solution of the form φ_c. However, the function $u = 0$ satisfies the differential equation and the initial condition $u(0) = 0$.

(c) With $u_0 > 0$ we have

$$c = \frac{1}{u_0} - 1 > -1.$$

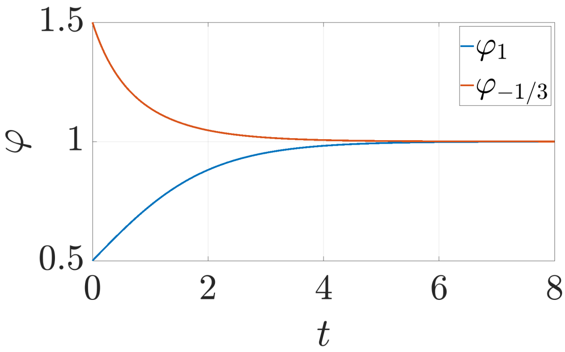
Since $\mathrm{e}^t > 1$ for $t > 0$, this means that $\mathrm{e}^t + c > 0$ for all $t > 0$.

(d) For $u_0 = \frac{1}{2}$ we have $c = 1$ and therefore

$$\varphi_1(t) = \frac{e^t}{e^t + 1}.$$

For $u_0 = \frac{3}{2}$ we have $c = -\frac{1}{3}$ and therefore

$$\varphi_{-\frac{1}{3}}(t) = \frac{e^t}{e^t - \frac{1}{3}}.$$



For arbitrary $u_0 > 0,\ u_0 \neq 1,$ and $c = (1 - u_0)/u_0$ we can write:

$$u(t) = \varphi_c(t) = \frac{\mathrm{e}^t}{e^t + c} = \frac{1}{1 + c\mathrm{e}^{-t}}$$

and from

$$\lim_{t\to\infty} \mathrm{e}^{-t} = 0$$

it follows

$$\lim_{t\to\infty} u(t) = 1,$$

independently on the initial value u_0! For $u_0 = 1$ we obtain $c = 0$ and thus $u(t) = 1$ for all $t \geq 0$.