Auditorium Exercise Sheet 4

Differential Equations I for Students of Engineering Sciences

Eleonora Ficola
Department of Mathematics, Universität Hamburg

Technische Universität Hamburg

Winter Semester 2024/2025

02.12.2024

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Linear homogeneous ODEs

Consider a linear, homogeneous differential equation of order $m \in \mathbb{N}$

$$A_m(t)u^{(m)}(t) + \cdots + A_2(t)u''(t) + A_1(t)u'(t) + A_0(t)u(t) = 0 \qquad (1)$$

with coefficients $A_k \in \mathrm{C}(I)$.

- There are exactly m linearly independent solutions of (1)
- If $u_1, u_2, \ldots, u_m$ are m linearly independent solutions of (1), then they build a **basis** of the space of solutions of (1) and $M := \{u_1, \ldots, u_m\}$ defines a **fundamental system** of the ODE (1)
- The **general solution** of the (homogeneous) ODE (1) is given by

$$u_h(t) := c_1 u_1(t) + c_2 u_2(t) + \cdots + c_m u_m(t), \qquad \text{with } c_k \in \mathbb{R}.$$

- Question: how to determine $u_1, \ldots, u_m$?

Linear hom. ODEs with constant coefficients

In case the coefficients in (1) are constants, we get:

$$a_m u^{(m)}(t) + a_{m-1} u^{(m-1)}(t) + \cdots + a_2 u''(t) + a_1 u'(t) + a_0 u(t) = 0 \qquad (2)$$

for $a_k \in \mathbb{R}$.

- We define the **characteristic polynomial** of (2) as

$$P(\lambda) := a_m \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

- If λ is a root (zero) of P, then the function $\boldsymbol{e^{\lambda t}}$ solves (2)
- If λ is a root of P with (algebraic) multiplicity $d \in \mathbb{N}$, then

$$\boldsymbol{e^{\lambda t},\ t \cdot e^{\lambda t},\ \ldots,\ t^{d-1} \cdot e^{\lambda t}}$$

  are d linearly independent solutions of (2)

Example 1

Write a fundamental system and the general solution of the ODE

$$u^{(4)}(t) - 5u'''(t) + 6u''(t) + 4u'(t) - 8u(t) = 0.$$

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$$u^{(4)}(t) - 5u'''(t) + 6u''(t) + 4u'(t) - 8u(t) = 0.$$

Characteristic polynomial: $P(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = (\lambda+1)(\lambda-2)^3$.

Roots of P are:

- $\lambda_1 = -1$, with multiplicity $d_1 = 1 \implies e^{-t}$ is a solution
- $\lambda_2 = 2$, with multiplicity $d_2 = 3 \implies e^{2t}, te^{2t}, t^2e^{2t}$ are other linearly independent solutions

Hence, a fundamental system is given by $M = \left\{e^{-t}, e^{2t}, te^{2t}, t^2e^{2t}\right\}$ and the general solution is

$$u_h(t) = c_1e^{-t} + c_2e^{2t} + c_3te^{2t} + c_4t^2e^{2t}, \qquad c_k \in \mathbb{R}.$$

Complex and real fundamental systems

- Recall: any polynomial of degree $m \in \mathbb{N}$ with real (or complex) coefficients has **exactly** m roots in $\mathbb{C}$ (counted with their multiplicity).
- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a root of the characteristic polynomial P associated to (2) with **real** coefficients, then its complex conjugate $\overline{\lambda}$ is still root of P, since

$$P\left(\overline{\lambda}\right) = \sum_{k=0}^{m} a_k \overline{\lambda}^k = \overline{\sum_{k=0}^{m} a_k \lambda^k} = \overline{P(\lambda)} \underset{\lambda \text{ root}}{=} 0.$$

  Meaning: complex solutions always appear in pairs of conjugates!

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Meaning: complex solutions always appear in pairs of conjugates!

Example 2

The ODE $u'' - 2u' + 5u = 0$ has characteristic polynomial $P(\lambda) = \lambda^2 - 2\lambda + 5$. Solve: $P(\lambda) = 0 \iff \lambda^2 - 2\lambda + 5 \iff \lambda = 1 \pm 2i$.

Then a (complex) fundamental system is given by $\left\{e^{(1+2i)t}, e^{(1-2i)t}\right\}$ and the general solution is: $u_h(t) = c_1 e^{(1+2i)t} + c_2 e^{(1-2i)t} = c_1 e^t e^{2it} + c_2 e^t e^{-2it}$.

Complex and real fundamental systems

Euler formula: for $\theta \in \mathbb{R}$, it is $e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta)$.

If $\lambda = a + ib \in \mathbb{C}$ $(a, b \in \mathbb{R},\ b \neq 0)$, $e^{\lambda t} = e^{(a+ib)t} = e^{at}\cos(bt) + ie^{at}\sin(bt)$.
Let $\overline{\lambda} = a - ib$ be its complex conjugate.

$$\Re(e^{\lambda t}) = e^{at}\cos(bt) = \frac{e^{\lambda t} + e^{\overline{\lambda} t}}{2} \rightsquigarrow \text{ real part of } e^{\lambda t}$$

$$\Im(e^{\lambda t}) = e^{at}\sin(bt) = \frac{e^{\lambda t} - e^{\overline{\lambda} t}}{2i} \rightsquigarrow \text{ imaginary part of } e^{\lambda t}$$

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If λ is root of the characteristic polynomial P of (2) (hence even $\overline{\lambda}$) $\implies$ $e^{\lambda t}, e^{\overline{\lambda} t}$ are two complex, linearly independent solutions of (2) $\implies$ $\Re(e^{\lambda t}), \Im(e^{\lambda t})$ are two real, linearly independent solutions of (2).

Example 3

Determine a **real** fundamental system and the (real) general solution of

$$u''(t) - 2u'(t) + 5u(t) = 0.$$

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From Example 2 we know that $e^{(1+2i)t}$ and $e^{(1-2i)t}$ are 2 lin. indep. *complex* solutions, which generate the *complex* fundamental system $M_{\mathbb{C}} := \left\{e^t e^{2it}, e^t e^{-2it}\right\}$.

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Determine a real fundamental system and the (real) general solution of

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From Example 2 we know that $e^{(1+2i)t}$ and $e^{(1-2i)t}$ are 2 lin. indep. *complex* solutions, which generate the *complex* fundamental system $M_{\mathbb{C}} := \left\{e^t e^{2it}, e^t e^{-2it}\right\}$.

Then $\mathfrak{R}(e^{(1+2i)t}) = e^t \cos(2t)$ and $\mathfrak{I}(e^{(1+2i)t}) = e^t \sin(2t)$ are lin. indep. *real* solutions, which generate the *real* fundamental system $M_{\mathbb{R}} := \{e^t \cos(2t), e^t \sin(2t)\}$.

The corresponding general solution of the ODE is then

$$u_h(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) \qquad c_1, c_2 \in \mathbb{R}.$$

Linear inhomogeneous ODEs

Consider a linear, inhomogeneous ODE of order $m \in \mathbb{N}$

$$A_m(t)u^{(m)}(t) + \cdots + A_2(t)u''(t) + A_1(t)u'(t) + A_0(t)u(t) = b(t) \qquad (3)$$

with coefficients $A_k, b \in \mathrm{C}(I)$.

- If u_h is the general solution of the corresponding homogeneous equation (1) and u_p is a particular solution of (3), then the general solution of (3) is given by

$$u(t) := u_h(t) + u_p(t)$$

- Question: how to determine u_p? *TO BE CONTINUED...*

Example 4

Determine the general solution of

$$u''(t) - 2u'(t) + 5u(t) = 13e^{-2t}. \tag{4}$$

Hint: Look up for particular solutions of the type $u_p(t) = Ce^{-2t}$, $C \in \mathbb{R}$.

Example 4

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Hint: Look up for particular solutions of the type $u_p(t) = Ce^{-2t}$, $C \in \mathbb{R}$.

The general solution of (4) is given by $u(t) = u_h(t) + u_p(t)$, for u_h general solution of the corresponding homogeneous ODE

$$u''(t) - 2u'(t) + 5u(t) = 0. \tag{5}$$

In Example 3 we computed $u_h(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$.

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In Example 3 we computed $u_h(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$.

Substitute the ansatz $u_p(t) = Ce^{-2t}$ to find C:
$u_p'(t) = -2Ce^{-2t} = -2u_p(t) \implies u_p''(t) = 4Ce^{-2t} = 4u_p(t)$. Thus:

$$(4C + (-2)(-2)C + 5C)e^{-2t} = 13e^{-2t} \implies C = 1 \implies u_p(t) = e^{-2t}$$

The general solution of (4) is:

$$u(t) = u_h(t) + u_p(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + e^{-2t}, \qquad \text{with } c_1, c_2 \in \mathbb{R}.$$

Linear homogeneous systems of first-order ODEs

Consider a linear, homogeneous system of $n \in \mathbb{N}$ first-order ODEs

$$\mathbf{u}'(t) = \mathbf{A}(t) \cdot \mathbf{u}(t) \tag{6}$$

with matrix of coefficients $\mathbf{A} \in \mathrm{C}(I, \mathbb{R}^{n\times n})$ and $\mathbf{u} \in \mathrm{C}^1(I, \mathbb{R}^n)$.

- There are exactly n linearly independent solutions (vectors!) of (6)
- If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in \mathrm{C}^1(I, \mathbb{R}^n)$ are n linearly independent solutions of (6), we say that they determine a **basis** (or a **fundamental system**) of the space of solutions of (6)
- The **general solution** of the homogeneous system (6) is given by

$$\mathbf{u}_h(t) := c_1\mathbf{u}_1(t) + c_2\mathbf{u}_2(t) + \cdots + c_n\mathbf{u}_n(t), \qquad \text{with } c_k \in \mathbb{R} \tag{7}$$

- If $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in \mathrm{C}^1(I, \mathbb{R}^n)$ are n solutions of the homogeneous system (6), the (function) matrix

$$\mathbf{W}(t) := \big(\mathbf{u}_1(t) \mid \quad \dots \quad \mid \mathbf{u}_n(t)\big) \quad \rightsquigarrow \mathbf{u}_k \text{ column vectors}$$

  is called a **solution matrix** of (6). In case $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ are even linearly independent, $\mathbf{W}$ is a **fundamental solution matrix** (or **Wronski matrix**) of (6).

  Note: $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ basis of $\mathrm{C}^1(I, \mathbb{R}^n) \iff \{\mathbf{u}_1(t), \dots, \mathbf{u}_n(t)\}$ basis of $\mathbb{R}^n$, for every $t \in I \iff \det(\mathbf{W}(t)) \neq 0$, for every $t \in I$.

- We may express the general solution (7) of (6) via the fundamental matrix:

$$\mathbf{u}_h(t) = \mathbf{W}(t) \cdot \mathbf{C}, \qquad \text{with } \mathbf{C} \in \mathbb{R}^n.$$

- Question: how to determine $\mathbf{u}_1, \dots, \mathbf{u}_m$?

Linear hom. systems with constant coefficients

In case the coefficients in (6) are constants, we get:

$$\mathbf{u}'(t) = \mathbf{A} \cdot \mathbf{u}(t) \tag{8}$$

with matrix of coefficients $\mathbf{A} \in \mathbb{R}^{n \times n}$.

- Compute the **characteristic polynomial** of (8) as

$$P(\lambda) := \det(\mathbf{A} - \lambda \mathbf{I}_n)$$

- If λ is an eigenvalue of $\mathbf{A}$ (i.e. a root of P) with corresponding eigenvector $\mathbf{v}$, then the function $\boldsymbol{u(t) := e^{\lambda t} v}$ is a solution of (8).

Example 5

For the homogeneous linear system with constant coefficients

$$\mathbf{u}'(t) = \begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{u}(t) \tag{9}$$

we compute the eigenvalues: $P(\lambda) = \det \begin{pmatrix} 4-\lambda & 5 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 4\lambda - 5 =$

$= (\lambda+1)(\lambda-5) = 0 \iff \lambda_1 = -1 \ \vee \ \lambda_2 = 5$, and thus the eigenvectors:

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- $\begin{pmatrix} 4-(-1) & 5 \\ 1 & -(-1) \end{pmatrix} \cdot \mathbf{v}_1 = \mathbf{0} \iff \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix} \cdot \mathbf{v}_1 = \mathbf{0}$, set for ex. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $\begin{pmatrix} 4-5 & 5 \\ 1 & -5 \end{pmatrix} \cdot \mathbf{v}_2 = \mathbf{0} \iff \begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix} \cdot \mathbf{v}_2 = \mathbf{0}$, set for ex. $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

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$= (\lambda+1)(\lambda-5) = 0 \iff \lambda_1 = -1 \lor \lambda_2 = 5$, and thus the eigenvectors:

- $\begin{pmatrix} 4-(-1) & 5 \\ 1 & -(-1) \end{pmatrix} \cdot \mathbf{v}_1 = \mathbf{0} \iff \begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix} \cdot \mathbf{v}_1 = \mathbf{0}$, set for ex. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $\begin{pmatrix} 4-5 & 5 \\ 1 & -5 \end{pmatrix} \cdot \mathbf{v}_2 = \mathbf{0} \iff \begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix} \cdot \mathbf{v}_2 = \mathbf{0}$, set for ex. $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Then two linearly independent solutions of (9) are: $\mathbf{u}_1(t) := \mathbf{v}_1 \cdot e^{\lambda_1 t} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$ and $\mathbf{u}_2(t) := \mathbf{v}_2 \cdot e^{\lambda_2 t} = \begin{pmatrix} 5e^{5t} \\ e^{5t} \end{pmatrix}$
and Wronski matrix $\mathbf{W}(t) = \begin{pmatrix} e^{-t} & 5e^{5t} \\ -e^{-t} & e^{5t} \end{pmatrix}$.

Linear hom. systems with constant coefficients

Whenever for the linear system

$$\mathbf{u}'(t) = \mathbf{A} \cdot \mathbf{u}(t), \quad \mathbf{A} \in \mathbb{R}^{n \times n} \tag{8}$$

we cannot find n eigenvectors of $\mathbf{A}$, we need to complete the basis...

For $\lambda \in \mathbb{C}$ eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ a corresponding eigenvector, we say that $\mathbf{w} \in \mathbb{C}^n$ is a **generalized eigenvector** (of rank 1) of $\mathbf{A}$ if $(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{w} = \mathbf{v}$.

In such a case, the functions $e^{\lambda t}\mathbf{v}$ and $e^{\lambda t}(\mathbf{w} + t\mathbf{v})$ are two linearly independent solutions of the homogeneous system (6).

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For $\lambda \in \mathbb{C}$ eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ a corresponding eigenvector, we say that $\mathbf{w} \in \mathbb{C}^n$ is a **generalized eigenvector** (of rank 1) of $\mathbf{A}$ if $(\mathbf{A} - \lambda \mathbf{I}_n)\mathbf{w} = \mathbf{v}$.

In such a case, the functions $\boldsymbol{e^{\lambda t}\mathbf{v}}$ and $\boldsymbol{e^{\lambda t}(\mathbf{w} + t\mathbf{v})}$ are two linearly independent solutions of the homogeneous system (6).

Remark: If $\lambda \in \mathbb{C}$ is root of the characteristic polynomial P of (8) with corresponding eigenvector $\mathbf{v}$, then $(\overline{\lambda}, \overline{\mathbf{v}})$ is another eigenvalue/vector pair $\Longrightarrow$ $e^{\lambda t}\mathbf{v}, e^{\overline{\lambda} t}\overline{\boldsymbol{v}}$ are two complex, linearly independent solutions of (8) $\Longrightarrow$ $\Re(e^{\lambda t}\mathbf{v}), \Im(e^{\lambda t}\mathbf{v})$ are two real, linearly independent solutions of (8).

Example 6

For the homogeneous linear system with constant coefficients

$$\mathbf{u}'(t) = \begin{pmatrix} 1 & 25 \\ -1 & -9 \end{pmatrix} \cdot \mathbf{u}(t) \tag{10}$$

we compute the eigenvalues: $P(\lambda) = \det \begin{pmatrix} 1-\lambda & 25 \\ -1 & -9-\lambda \end{pmatrix} = \lambda^2 + 8\lambda + 16 =$
$= (\lambda + 4)^2 = 0 \iff \lambda = -4$ with algebraic multiplicity $d = 2$.

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$= (\lambda+4)^2 = 0 \iff \lambda = -4$ with algebraic multiplicity $d = 2$.

- Eigenvector(s):

$$\begin{pmatrix} 1-(-4) & 25 \\ -1 & -9-(-4) \end{pmatrix} \cdot \mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 5 & 25 \\ -1 & -5 \end{pmatrix} \cdot \mathbf{v} = \mathbf{0} \rightsquigarrow \mathbf{v} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$$

$\Longrightarrow$ one solution is $\mathbf{u}_1(t) = \mathbf{v} \cdot e^{\lambda t} = \begin{pmatrix} -5e^{-4t} \\ e^{-4t} \end{pmatrix}$.

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$= (\lambda+4)^2 = 0 \iff \lambda = -4$ with algebraic multiplicity $d = 2$.

- We still need $n-1 = 1$ element of the basis: compute a generalized eigenvector $\mathbf{w}$: $(\mathbf{A} - \lambda \mathbf{I}_2)\mathbf{w} = \mathbf{v} \iff \begin{pmatrix} 5 & 25 \\ -1 & -5 \end{pmatrix} \cdot \mathbf{w} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$, set for ex. $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Another lin. indep. sol of (10) is $\mathbf{u}_2(t) = e^{\lambda t}(\mathbf{w} + t\mathbf{v}) = \begin{pmatrix} -(1+5t)e^{-4t} \\ te^{-4t} \end{pmatrix}$.

Linear inhomogeneous systems of first-order ODEs

Consider a linear, inhomogeneous system of $n \in \mathbb{N}$ first-order ODEs

$$\mathbf{u}'(t) = \mathbf{A}(t) \cdot \mathbf{u}(t) + \mathbf{b}(t) \tag{11}$$

with matrix $\mathbf{A} \in \mathrm{C}(I, \mathbb{R}^{n \times n})$, inhomogeneity $\mathbf{b} \in \mathrm{C}(I, \mathbb{R})$ and $\mathbf{u} \in \mathrm{C}^1(I, \mathbb{R}^n)$.

- If $\mathbf{u}_h$ is the general solution of the corresponding homogeneous system and $\mathbf{u}_p$ is a particular solution of (11), then the general solution of (11) is given by

$$\mathbf{u}(t) := \mathbf{u}_h(t) + \mathbf{u}_p(t)$$

- Question: how to determine $\mathbf{u}_p$? *TO BE CONTINUED...*

From ODEs to systems

Given a (scalar) ODE of any order $n \in \mathbb{N}$ (for example, in explicit form)

$$u^{(n)} = f(t, u, u', u'', \dots, u^{(n-1)}) \qquad \text{for } u : I \to \mathbb{R}, \tag{12}$$

we introduce the functions $u_1, u_2, \dots, u_n : I \to \mathbb{R}$ defined as

$$\boldsymbol{u_1 := u,\ u_2 := u_1{}' = u',\ u_3 := u_2{}' = u'', \dots, u_n := (u_{n-1})' = u^{(n\text{-}1)}},$$

from which $\boldsymbol{u_n{}' = u^{(n)}}$. We rewrite the ODE in (12) as

$$u_n{}' = f(t, u_1, u_2, u_3, \dots, u_n)$$

Taking into account the definitions of $u_1, \dots, u_n$, we obtained the **system of n ODEs** of first-order:

$$\begin{cases} u_1{}' = u_2 \\ u_2{}' = u_3 \\ \vdots \qquad \vdots \\ (u_{n-1})' = u_n \\ (u_n)' = f(t, u_1, u_2, u_3, \dots, u_n) \end{cases}$$

Linear ODEs of n-th order as linear systems of n first-order ODEs

Specifically, if the ODE in (12) is linear of order n, i.e. in the explicit form

$$u^{(n)} = b - a_0 u - a_1 u' - \ldots - a_{n-1} u^{(n-1)} \tag{13}$$

with $a_0, a_1, \ldots, a_{n-1}, b : I \to \mathbb{R}$ functions on $I \subseteq \mathbb{R}$, then u is solution of (13) if and only if $(u_1, u_2, \ldots, u_n)$ solves the system

$$\begin{cases} {u_1}' = u_2 \\ {u_2}' = u_3 \\ \vdots \qquad \vdots \\ (u_{n-1})' = u_n \\ {u_n}' = b - a_0 u - a_1 u' - \ldots - a_{n-1} u^{(n-1)} \end{cases} \tag{14}$$

Let $\mathbf{u}(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ and $\mathbf{B}(t) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix}$ be vectors of n components,

$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ -a_0(t) & -a_1(t) & \dots & -a_{n-1}(t) \end{pmatrix}$ matrix of order n.

Rewrite (14) as

$$\mathbf{u}'(t) = \mathbf{A}(t) \cdot \mathbf{u}(t) + \mathbf{B}(t).$$

- The following bijection holds:

$$\left\{\begin{matrix}\textbf{linear } \boldsymbol{n}\textbf{-th order ODEs} \\ \textbf{in explicit form}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\textbf{linear systems of } \boldsymbol{n} \textbf{ ODEs} \\ \textbf{of order 1}\end{matrix}\right\}$$

$$u^{(n)}(t) = b(t) - \sum_{i=0}^{n-1} a_i(t) u^{(i)}(t) \longleftrightarrow \mathbf{u}'(t) = \mathbf{A}(t) \cdot \mathbf{u}(t) + \mathbf{B}(t)$$

Example 7

Rewrite the following IVP of a third order ODE

$$\begin{cases} 3u''' + 4t\cos(2t)u' - e^t u + 6t = 12, \ \ t > 5; \\ u(5) = -1, \ \ u'(5) = 0, \ \ u''(5) = 2 \end{cases} \tag{15}$$

as an initial value problem for a first-order system.

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as an initial value problem for a first-order system.
Set $u_1 := u$, $u_2 := {u_1}' = u'$, $u_3 := {u_2}' = u'' \implies {u_3}' = u'''$.

Substituting into (15) returns : $3{u_3}' + 4t\cos(2t)u_2 - e^t u_1 + 6t = 12$
$\implies {u_3}' = (-4t\cos(2t)u_2 + e^t u_1 + 12 - 6t)/3$.

$$\underbrace{\begin{pmatrix} {u_1}' \\ {u_2}' \\ {u_3}' \end{pmatrix}}_{\boldsymbol{u}'} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^t/3 & -4t\cos(2t)/3 & 0 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{\mathbf{u}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 4 - 2t \end{pmatrix}}_{\mathbf{B}}$$

with $\mathbf{u}(5) = \begin{pmatrix} u_1(5) \\ u_2(5) \\ u_3(5) \end{pmatrix} = \begin{pmatrix} u(5) \\ u'(5) \\ u''(5) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exercise

Consider the following linear homogeneous ODE in $u = u(t)$:

$$u^{(5)} - 4u^{(4)} + 9u''' - 18u'' + 20u' - 8u = 0. \tag{16}$$

with initial conditions

$$u(0) = 1,\ u'(0) = 1,\ u''(0) = 0,\ u'''(0) = -3,\ u^{(4)}(0) = -10$$

(i) Determine a real fundamental system and the general solution of (16).

(ii) Write (16) in explicit form in the domain $\{t \in \mathbb{R} : t > 0\}$.

(iii) Rewrite (16) as a system of first-order ODEs.

(iv) Find a Wronski matrix and the general solution of the system in (iii). Compare it with the result of (i).

(v) Solve the corresponding initial value problem with the prescribed values.

AUDITORIUM EXERCISE CLASS 4

EXERCISE:

lin. hom. ODE of order $m=5$ with constant coefficients

(16) $u^{(5)} - 4u^{(4)} + 9u''' - 18u'' + 20u' - 8u = 0, \quad u = u(t)$

(i) characteristic polynomial: $P(\lambda) := \lambda^5 - 4\lambda^4 + 9\lambda^3 - 18\lambda^2 + 20\lambda - 8 \overset{\text{set}}{=} 0$

$P(\lambda) = (\lambda-1)^2 \cdot (\lambda-2)(\lambda-2i)(\lambda+2i)$

$e^{2it} = \cos(2t) + i\sin(2t)$

$\lambda_1 = 1$, mult. $d_1 = 2 \Rightarrow u_1(t) = e^{\lambda_1 t} = e^t$; $u_2(t) = t \cdot e^{\lambda_1 t} = te^t$

$\lambda_2 = 2$, mult. $d_2 = 1 \Rightarrow u_3(t) = e^{\lambda_2 t} = e^{2t}$

$\lambda_{3/4} = \pm 2i$, mult $d_{3/4} = 1 \Rightarrow u_4(t) := \mathrm{Re}(e^{\lambda_3 t}) = \mathrm{Re}(e^{2it}) = \cos(2t)$

real solutions

$u_5(t) := \mathrm{Im}(e^{\lambda_3 t}) = \mathrm{Im}(e^{2it}) = \sin(2t)$

General (real) solution of (16): $u_h(t) := c_1 e^t + c_2 t e^t + c_3 e^{2t} + c_4 \cos(2t) + c_5 \sin(2t)$, $c_i \in \mathbb{R}$

$M = \{e^t, te^t, e^{2t}, \cos(2t), \sin(2t)\} \leftarrow$ 5 lin. indep sol, M: real fundamental system

(ii) $u^{(5)} = 8u - 20u' + 18u'' - 9u''' + 4u^{(4)} \rightarrow$ explicit form of (16)

(iii) Set $u_1 := u$, $u_2 := u' = u_1'$, $u_3 := u'' = u_1'' = u_2'$, $u_4 := u_3'$, $u_5 := u_4'$

$\Rightarrow u_5' = u^{(5)} = 8u_1 - 20u_2 + 18u_3 - 9u_4 + 4u_5$

$$\mathcal{U}'(t) := \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \\ u_4'(t) \\ u_5'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 8 & -20 & 18 & -9 & 4 \end{pmatrix}}_{=A} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} (+\,0)$$

because (16) is homogeneous!

(iv) Characteristic polynomial of A: $\tilde{P}(\lambda) = \det(A - \lambda I_5) = \begin{vmatrix} -\lambda & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 1 \\ 8 & -20 & 18 & -9 & 4-\lambda \end{vmatrix} = 0 \Rightarrow$ computations

$\Rightarrow \tilde{P}(\lambda) = -\lambda^5 + 4\lambda^4 - 9\lambda^3 + 18\lambda^2 - 20\lambda + 8$

Same roots as in (i): $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_{3/4} = \pm 2i$ $\leftarrow$ Eigenvalues of A

We need the corresponding eigenvectors v...

$\boxed{\lambda_1 = 1}$

$(A - \lambda_1 I_5) \cdot v_1 = 0$ $\qquad$ $u(t) = v \cdot e^{\lambda t}$ then is a sol. of (16)

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 8 & -20 & 18 & -9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_2 = 1 + x_1 \\ x_3 = 1 + x_2 = 2 + x_1 \\ x_4 = 1 + x_3 = 3 + x_1 \\ x_5 = 1 + x_4 = 4 + x_1 \end{array}$$

For example, let $x_1 = 0$: $v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\leftarrow$ 1st eigenvector

We then define $\left[u_1(t) := v_1 \cdot e^{\lambda_1 t} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \cdot e^t \right] \rightarrow$ one sol. of (16)

But λ_1 has multiplicity 2: we need a second eigenvector... TO BE CONTINUED

↳ in order to define a second solution linearly indep. from u_1!

$\boxed{\lambda_2 = 2}$ Set eigenvector: $(A-\lambda_2 I_5)\cdot v_2 = 0 \Rightarrow \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 8 & -20 & 18 & -9 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 = 2x_1 \\ x_3 = 2x_2 \\ x_4 = 2x_3 \\ x_5 = 2x_4 \end{matrix}$ Set $x_1 = 1$:

$v_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{pmatrix} \rightarrow$ eigenvector of λ_2

We then define $\left[u_2(t) := v_2 \cdot e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \\ 16 \end{pmatrix} \cdot e^{2t} \right] \rightarrow$ <u>one</u> sol. of (16)

$\boxed{\lambda_{3/4} = \pm 2i}$ Set eigenvectors: $(A-\lambda_3 I_5)\cdot v_3 = 0 \Rightarrow \begin{pmatrix} -2i & 1 & 0 & 0 & 0 \\ 0 & -2i & 1 & 0 & 0 \\ 0 & 0 & -2i & 1 & 0 \\ 8 & -20 & 18 & -9 & 4-2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 = 2i x_1 \\ x_3 = 2i x_2 = -4x_1 \\ x_4 = 2i x_3 = -8i x_1 \\ x_5 = 2i x_4 = 16 x_1 \end{matrix}$

Set $x_1 = 1$: $v_3 = \begin{pmatrix} 1 \\ 2i \\ -4 \\ -8i \\ 16 \end{pmatrix} \Rightarrow \left.\begin{matrix} \hat{u}_3(t) := v_3 \cdot e^{\lambda_3 t} \\ \hat{u}_4(t) := \bar{v}_3 \cdot e^{\bar{\lambda}_3 t} = v_4 \cdot e^{\lambda_4 t} \end{matrix}\right\}$ are complex solutions: the corresponding <u>real</u> solutions are:

$u_3(t) := \mathrm{Re}(\hat{u}_3(t))$, and computing:

$u_4(t) := \mathrm{Im}(\hat{u}_3(t))$

"Real part" "Imaginary part"

$$\hat{u}_3(t) = e^{2it}\begin{pmatrix} 1 \\ 2i \\ -4 \\ 8i \\ 16 \end{pmatrix} = \begin{pmatrix} \cos(2t) + i\sin(2t) \\ -2\sin(2t) + i\,(2\cos(2t)) \\ -4\cos(2t) + i\,(-4\sin(2t)) \\ 8\sin(2t) + i\,(-8\cos(2t)) \\ 16\cos(2t) + i\,(16\sin(2t)) \end{pmatrix} \Rightarrow u_3(t) = \begin{pmatrix} \cos(2t) \\ -2\sin(2t) \\ -4\cos(2t) \\ 8\sin(2t) \\ 16\cos(2t) \end{pmatrix}, \quad u_4(t) = \begin{pmatrix} \sin(2t) \\ 2\cos(2t) \\ -4\sin(2t) \\ -8\cos(2t) \\ 16\sin(2t) \end{pmatrix}$$

other lin. indep. sol. of (16)

Remaining part of (iv) and (v) to be completed (need 1st to compute the generalized EV...)