Auditorium Exercise Sheet 3

Differential Equations I for Students of Engineering Sciences

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Similarity ODE

A first-order ODE of the type

$$u'(t) = f\left(\frac{u(t)}{t}\right) \tag{1}$$

for some function $f: I \to \mathbb{R}$ is called a **similarity equation**.

It can be solved by applying a change of variables: setting $\boldsymbol{y(t) := \frac{u(t)}{t}}$, it is $u'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(t \cdot y(t)) = y(t) + ty'(t)$. Thus (1) becomes

$$y(t) + ty'(t) = f(y(t)) \to \text{separable variables ODE in } y$$

Finally, substitute back to find u general sol. of (1).

Example 1

Find the general solution of the ODE

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Rewriting the equation it is $u' = \left(\frac{t}{u}\right)^3 + \frac{u}{t}$, which is a similarity equation with $f(y) = y + y^{-3}$ for $y(t) := \frac{u}{t}$. Apply the substitution:

$$y(t) := \frac{u}{t} \implies u(t) = ty(t) \implies u'(t) = y(t) + ty'(t).$$

We obtain the separable variable ODE: $\cancel{y} + ty' = \cancel{y} + y^{-3}$

$$\int y^3 \mathrm{d}y = \int y^3 y' \mathrm{d}t = \int \frac{1}{t} \mathrm{d}t$$
$$y^4(t) = 4 \ln |t| + C \implies y(t) = \pm(\ln(t^4) + C)^{1/4}, \ C \in \mathbb{R}.$$

Substitute back: $u(t) = ty(t) = \pm t(\ln(t^4) + C)^{1/4}$, $C \in \mathbb{R}$.

Riccati equation*

A first order (non-linear) equation of the form

$$u'(t) = a(t)u(t) + b(t)u^2(t) + c(t), \qquad \text{with } a, b, c \in \mathrm{C}(I) \tag{2}$$

is called **Riccati differential equation**.

Suppose we know a **particular solution u_p** of (2).
Then the function $v(t) := u(t) - u_p(t)$ solves the Bernoulli ODE
$v'(t) = [a(t) + 2b(t)u_p(t)]v(t) + b(t)v^2(t)$.

Thus setting $\boldsymbol{y(t) := v^{-1}(t) = \frac{1}{u(t)-u_p(t)}}$, we find the first-order linear ODE

$$y'(t) = -y(t)[a(t) + 2b(t)u_p(t)] - b(t) \tag{3}$$

to be solved in y. Finally, substitute back to find u general sol. of (2).

* Studied by the Venetian mathematician Jacopo Riccati (1676-1754)

Example 2

Find the general solution of the ODE $u' = -u^2 + \frac{2}{t^2}$, for $u = u(t)$ and $t > 0$.

It is a Riccati equation with $a(t) = 0$, $b(t) = -1$ and $c(t) = \frac{2}{t^2}$. Taking $u_p(t) = \frac{k}{t}$, $k \in \mathbb{R}$ as Ansatz for a particular solution, find the appropriate k.

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By substitution we see that $u_p(t) = -1/t$ is a solution.
Let $y(t) := \frac{1}{u(t) - u_p(t)}$, and applying (3) the ODE becomes:

$$y'(t) = -y(t)[0 + 2(-1)(-1/t)] + 1 = 1 - 2y(t)/t$$

Solving the latter in y yields $y(t) = \frac{t^3 + C}{3t^2}$, for $C \in \mathbb{R}$.

Returning to u we obtain:

$$u(t) = u_p(t) + \frac{1}{y(t)} = -\frac{1}{t} + \frac{3t^2}{t^3 + C} = \frac{2t^3 - C}{t(t^3 + C)} \quad \vee \quad u(t) = u_p(t) = -1/t$$

Euler-Cauchy equation*

A special type of linear homogeneous ODE of arbitrary order m

$$a_m t^m u^{(m)}(t) + \cdots + a_2 t^2 u''(t) + a_1 t u'(t) + a_0 u(t) = 0$$

with $a_i \in \mathbb{R}$ and $a_m \neq 0$ is called **Euler-Cauchy differential equation**.
In particular, we consider the second-order case:

$$a_2 t^2 u''(t) + a_1 t u'(t) + a_0 u(t) = 0 \tag{4}$$

*Named after the Swiss mathematician Leonhard Euler (1707–1783) and the French mathematician/engineer Augustin-Louis Cauchy (1789-1857).

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When solving in $t > 0$, we can apply the substitution $s := \ln(t)$, from which $t = e^s$, and introducing the function $\boldsymbol{y(s) := u(e^s) = u(t)}$ we obtain:

$$\begin{aligned} y'(s) &= tu'(t) \\ y''(s) &= t^2 u''(t) + tu'(t) \implies t^2 u''(t) = y''(s) - y'(s) \end{aligned}$$

Inserting into (4) returns the following second-order linear homogeneous ODE with constant coefficients:

$$a_2 \left(y''(s) - y'(s) \right) + a_1 y'(s) + a_0 y(s) = 0$$

*Named after the Swiss mathematician Leonhard Euler (1707–1783) and the French mathematician/engineer Augustin-Louis Cauchy (1789-1857).

Exact differential equations

Let $D \subseteq \mathbb{R}^2$ open. A first order ODE of the form

$$f(t, u(t)) + g(t, u(t)) \cdot u'(t) = 0 \tag{5}$$

is called **exact** in D if there exists a C^1 **potential** $\psi : D \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial \psi}{\partial t}(t, u) = f(t, u) \\ \frac{\partial \psi}{\partial u}(t, u) = g(t, u), \end{cases}$$

for all $(t, u) \in D$. In such a case, $u = u(t)$ solves (5) if and only if

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t, u(t)) = \frac{\partial \psi}{\partial t}(t, u(t)) + \frac{\partial \psi}{\partial u}(t, u)\frac{\mathrm{d}u}{\mathrm{d}t} = f(t, u) + g(t, u) \cdot u'(t) = 0$$
$$\iff \psi(t, u(t)) = K, \text{ for every } K \in \mathbb{R}.$$

Necessary and sufficient conditions to exact ODEs

Determining if an ODE of the kind

$$f(t, u(t)) + g(t, u(t)) \cdot u'(t) = 0 \tag{5}$$

is exact by applying the definition may not be immediate.
For this reason, we make use of the following criterion:

Theorem (integrability criterion for exact ODEs)

If f and g are $\mathrm{C}^1(D)$ with $D \subseteq \mathbb{R}^2$ simply connected, then:

$$(5) \text{ is exact in } D \iff \frac{\partial f}{\partial u}(t, u) = \frac{\partial g}{\partial t}(t, u), \quad \text{for all } (t, u) \in D.$$

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Example: the differential equation $\mathbf{2tu(t)} + (\boldsymbol{t}^2 + \boldsymbol{u}^2(\boldsymbol{t}) + 3\boldsymbol{u}(\boldsymbol{t}))u'(t) = 0$ is exact in $\mathbb{R}^2$, since $\frac{\partial \mathbf{f}}{\partial u}(t, u) = 2t = \frac{\partial \mathbf{g}}{\partial t}(t, u)$ for every $(t, u) \in \mathbb{R}^2$.

Integrating factor

In case the ODE

$$f(t, u(t)) + g(t, u(t)) \cdot u'(t) = 0 \tag{5}$$

is NOT exact, we look for an equivalent equation (i.e. with same solution) multiplying (5) by an **integrating factor** $h = h(t, u(t)) \neq 0$, that is

$$h(t, u(t)) \cdot f(t, u(t)) + h(t, u(t)) \cdot g(t, u(t)) \cdot u'(t) = 0, \tag{6}$$

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$$h(t, u(t)) \cdot f(t, u(t)) + h(t, u(t)) \cdot g(t, u(t)) \cdot u'(t) = 0, \tag{6}$$

and setting the integrability criterion for $h \cdot f$, $h \cdot g$:

$$\frac{\partial hf}{\partial u}(t, u) = \frac{\partial hg}{\partial t}(t, u) \implies \text{(6) exact.}$$

To determine h, look first for Ansatz of the type $h = h(t)$ or $h = h(u)$. Once h is obtained, solve the exact ODE (6) by finding a potential and setting $\psi(t, u(t)) = K$, with $K \in \mathbb{R}$ arbitrary.

Exercises

Exercise 1. Solve the following Euler-Cauchy differential equation

$$2t^2 \cdot u'' + tu' - 3u = 0, \quad t > 0.$$

Hint: linear, homogeneous ODEs in s with constant coefficients admit solutions of the type $e^{\lambda s}$...

Exercise 2. Solve by substitution the following similarity differential equations.

$$(i)\ u' = \frac{u + 2t}{t},\ t \neq 0$$

$$(ii)\ \dot{x} = \frac{t^2 + x^2}{tx},\ t \neq 0$$

Exercise 3. Solve the following Riccati differential equations. Use the hints to find a particular solution first.

(i) $u' + 6u^2 = 1/t^2, \ t > 0$

with particular sol. $u_p(t) := \dfrac{\alpha}{t} + \beta$

(ii) $u' + 4t = tu^2$

with particular sol. a constant

(iii) $x^3 u' + x^2 u - u^2 = 2x^4, \ x > 1$

with particular sol. a polynomial of degree 2 in x

Exercise 4. For any of the following differential equations, determine:

- Which one are exact.
- For each exact equation, compute a corresponding potential ψ.
- For each non-exact equation, determine an integrating factor $h = h(t,u)$ such that the new ODE is exact.
- Determine the general solution of the exact ODEs by solving the (algebraic) level set equation for the potential $\psi(t,u(t)) = K$.

$$(i)\ 2tu + (t^2+3)u' = 0$$

$$(ii)\ u' + 2tu = e^{t-t^2}$$

$$(iii)\ u + (x-1)u' = -2x,\ x > 1$$

$$(iv)\ 3x^2 + u^2 + 2u(1+x)u' = 0,\ x \in (1,10)$$

$$(v)\ -u\cos(t) = u'(\sin(t) + \sin(u) + u\cos(u))$$

$$(vi)\ u^2 + (tu+1)u' = 0$$

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Exercise 4

(i) $\underset{f}{2t\cdot u}+\underset{g}{(t^2+3)}u'=0$

$$\left[\begin{array}{l} f+g\cdot u'=0 \\ \text{EXACT iff } \dfrac{\partial f}{\partial u}=\dfrac{\partial g}{\partial t} \end{array}\right. \xrightarrow{\text{if it is exact!}} \text{POTENTIAL } \Psi=\Psi(t,u): \begin{cases} \frac{\partial \Psi}{\partial t}=f \\ \frac{\partial \Psi}{\partial u}=g \end{cases}$$

$\dfrac{\partial f}{\partial u}=2t = \dfrac{\partial g}{\partial t}=2t+0$ ✓

$\Psi=\Psi(t,u)$ s.t. $\begin{cases} \dfrac{\partial \Psi}{\partial t}=2t\cdot u \\ \dfrac{\partial \Psi}{\partial u}=t^2+3 \end{cases}$

$\rightsquigarrow \left[\Psi(t,u)=\int \dfrac{\partial \Psi}{\partial t}dt=\int 2tu\,dt=t^2u+c(u)\right]$

constant in t!

From the second equation: $t^2+3=\dfrac{\partial \Psi}{\partial u}=\dfrac{\partial}{\partial u}(t^2u+c(u))=t^2+c'(u) \Rightarrow \cancel{t^2}+3=\cancel{t^2}+c'(u) \Rightarrow$

$\Rightarrow c'(u)=3 \Rightarrow c(u)=3u+c$, $\mathbb{R}$!

Altogether: $\left[\Psi_c(t,u)=t^2u+3u+c,\ c\in\mathbb{R}\right]$ potential for the ODE

Recall: $\dfrac{d\Psi}{dt}(t,u(t))=0 \rightsquigarrow \Psi(t,u(t))=K \to$ constant on $u=u(t)$ solutions!

WLOG $c=0$: $\Psi(t,u)=t^2u+3u$

$\hookrightarrow$ I just need one potential

For u sol of ODE, we get: $\Psi(t,u)=K$

$t^2u+3u=K$

$u(t^2+3)=K \Rightarrow \boxed{u(t)=\dfrac{K}{t^2+3},\ K\in\mathbb{R}}$ general sol of the (exact) ODE

(ii) $u' + 2t\cdot u = e^{t-t^2} \to \underbrace{\left(2t\cdot u - e^{t-t^2}\right)}_{f} + \underbrace{1}_{g}\cdot u' = 0$

$\frac{\partial f}{\partial u} = 2t - 0 = 2t$

$\frac{\partial g}{\partial t} = 0$ $\neq$ NOT EXACT! Look for integrating factor $h = h(t, u(t))$

$\underbrace{(2tu - e^{t-t^2})}_{f} + u' = 0$, $g = 1$ NOT exact! Multiply by h [assume $h = h(t)$, $\neq 0$]: $\underbrace{(2tu - e^{t-t^2})h}_{F} + \underbrace{h}_{G}u' = 0$

It's exact if $\frac{\partial F}{\partial u} = \frac{\partial G}{\partial t}$ | $\frac{\partial G}{\partial t} = \frac{\partial h}{\partial t} = h'(t)$, $\frac{\partial F}{\partial u} = h(2t - 0) = 2th \Rightarrow$

Set $h'(t) = 2t\cdot h(t) \Rightarrow \ln|h(t)| = \int \frac{h'(t)}{h(t)}dt = \int 2t\,dt = t^2 + c \Rightarrow |h(t)| = e^c\cdot e^{t^2}$

linear, hom... $h(t) \neq 0$ sep. var.

$h_c(t) = \pm e^c e^{t^2} = c_1 e^{t^2}$, $c_1 \in \mathbb{R}$

$\pm e^c \mapsto c_1$

I just need 1 integrating factor: $h(t) = e^{t^2}$ $(c_1 = 1)$

We got: $\left(2tu - e^{t-t^2}\right)e^{t^2} + e^{t^2}u' = 0$ is exact!

Compute potential Ψ s.t

$$\begin{cases} \frac{\partial \Psi}{\partial t} = F = (2tu - e^{t-t^2})e^{t^2} = 2tue^{t^2} - e^t \\ \frac{\partial \Psi}{\partial u} = G = e^{t^2} \Rightarrow \left[\Psi(t,u) = \int \frac{\partial \Psi}{\partial u}du = \int e^{t^2}du = e^{t^2}\cdot u + c(t)\right] \end{cases}$$

To find $c(t)$ we employ the second equation: $\cancel{2tue^{t^2}} - e^t = \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t}\left(e^{t^2}\cdot u + c(t)\right) = \cancel{2te^{t^2}u} + c'(t)$

$\Rightarrow c'(t) = -e^t \Rightarrow c(t) = -\int e^t dt = -e^t + c$, $c \in \mathbb{R}$

Potentials are of the kind: $\Psi_c(t,u) = e^{t^2}\cdot u + c - e^t$, and we can set $c = 0$: $\left[\Psi(t,u) = e^{t^2}u - e^t\right]$

If $u(t)$ sol. of the ODE, then: $\Psi(t, u(t)) = K$, $K \in \mathbb{R} \Rightarrow e^{t^2}\cdot u - e^t = K$

$u(t) = (K + e^t)e^{-t^2}$, $K \in \mathbb{R}$

is the general sol. of the (original) ODE

EXERCISE 1

$\boxed{2t^2 \cdot u''} + tu' - 3u = 0$, $t>0$ $\rightarrow$ Euler-Cauchy ODE

Set $y(s) := u(e^s)$, $t =: e^s$ $\rightsquigarrow$ $y'(s) = t \cdot u'(t)$
$\boxed{t^2 \cdot u''(t)} = y''(s) - y'(s)$

Substitute:
$2(y''(s) - y'(s)) + y'(t) - 3y(s) = 0$

(*) $\boxed{2y'' - y' - 3y = 0}$ $\leftarrow$ is a 2^{nd}-order, linear ODE with constant coefficients! Solutions are of the type $[y(s) = e^{\lambda s}]$, λ to be det!

we will see this at a later point in the theory...

$y'(s) = \lambda e^{\lambda s} = \lambda y$
$y'' = \lambda^2 \cdot e^{\lambda s} = \lambda^2 \cdot y$

Substitute: $2\lambda^2 \cdot y(s) - \lambda y(s) - 3y(s) = 0 \Rightarrow y(s)(2\lambda^2 - \lambda - 3) = 0 \underset{y(s) = e^{\lambda s} > 0}{\Rightarrow}$

$\Rightarrow 2\lambda^2 - \lambda - 3 = 0 \rightarrow$ characteristic polynomial of the new ODE (*)

$\lambda_{1/2} = \frac{1 \pm 5}{4}$ $\nearrow \lambda_1 = 3/2$, $\searrow \lambda_2 = -1$

So: $y_1(s) = e^{3/2 s} = (e^s)^{3/2}$
$\vee$
$y_2(s) = e^{-s} = (e^s)^{-1}$

The equation (*) is linear: a linear combination of solutions is still a solution: the general sol. of (*) is given by

$$y(s) = c_1 \cdot (\underbrace{e^s}_{t})^{3/2} + c_2 \cdot (e^s)^{-1}, \quad c_1, c_2 \in \mathbb{R}$$

Substitute back: $u(t) = c_1 t^{3/2} + c_2 t^{-1}$, $c_1, c_2 \in \mathbb{R}$ $\longrightarrow$ this is the general sol. of our original ODE!