Auditorium Exercise Sheet 2

Differential Equations I for Students of Engineering Sciences

Eleonora Ficola
Department of Mathematics, Universität Hamburg

Technische Universität Hamburg

Winter Semester 2024/2025

04.11.2024

Table of contents

Separable variables ODE

A first order ODE of the form

$$u'(t) = g(t) \cdot h(u(t))$$

with g, h continuous real functions is called a **separable variables** ODE.

In case $h(u(t)) \neq 0$ for all t, we can solve it dividing both sides by $h(u)$ and then integrating with respect to the independent variable t:

$$\int g(t)\mathrm{d}t = \int \frac{u'(t)}{h(u(t))}\mathrm{d}t \underset{u \mapsto u(t)}{=} \int \frac{\mathrm{d}u}{h(u)}$$

After integrating, explicit u.

Example 1

Solve the ODE $\quad u'(t) = 2tu^3(t) \quad$ under initial condition $u(0) = 1$.
Which is the largest interval in which the solution is defined?

Example 1

Solve the ODE $\quad u'(t) = 2tu^3(t) \quad$ under initial condition $u(0) = 1$.
Which is the largest interval in which the solution is defined?

It is a separable variable ODE with $g(t) := 2t$ and $h(u) := u^3$. Notice that $u \equiv 0$ is a solution of the equation, but it does NOT satisfy the initial condition. Suppose then $u \neq 0$ and compute:

$$\int \frac{\mathrm{d}u}{u^3} \underset{u \mapsto u(t)}{=} \int \frac{u'(t)}{u^3(t)} \mathrm{d}t = \int 2t \mathrm{d}t$$

$$-\frac{1}{2u^2} = t^2 + C_1 \implies u(t) = \pm \frac{1}{\sqrt{C_2 - 2t^2}} \rightarrow \text{gen. sol. of the ODE}$$

Employ now the initial condition:

$$1 = u(0) = \pm \frac{1}{\sqrt{C_2 - 2 \cdot 0^2}} \implies C_2 = 1 \text{ and } u(t) = \frac{1}{\sqrt{1 - 2t^2}}.$$

Example 1

Solve the ODE $u'(t) = 2tu^3(t)$ under initial condition $u(0) = 1$.
Which is the largest interval in which the solution is defined?

The solution $u(t) = \frac{1}{\sqrt{1-2t^2}}$ is well-defined where $1 - 2t^2 > 0$, namely in the interval $I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

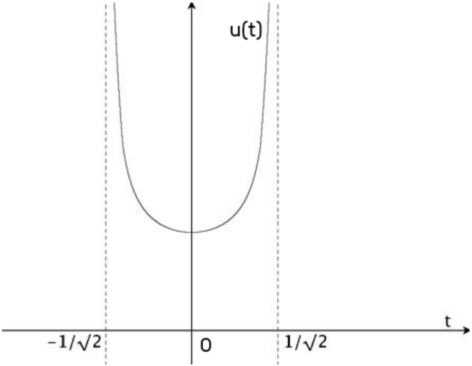


Figure: Graph of $u(t) = \frac{1}{\sqrt{1-2t^2}}$

Bernoulli equation[*]

A first order (non-linear) equation of the form

$$u'(t) = a(t)u(t) + b(t)u(t)^{\alpha}, \quad a, b \in \mathrm{C}(I),\ \alpha \in \mathbb{R} \setminus \{0, 1\} \qquad (1)$$
$$\text{(in gen. for } u > 0 \text{ if } \alpha \notin \mathbb{N}\text{)}$$

is called **Bernoulli differential equation**.

With the substitution $\boldsymbol{y(t) = u^{1-\alpha}(t)}$, it is $y'(t) = (1-\alpha)u'(t)u(t)^{-\alpha}$ and dividing the equation (1) by u^{α} we get

$$y'(t) = (1-\alpha)[a(t)y(t) + b(t)] \to \text{first-order, linear ODE } \textbf{in } \boldsymbol{y}$$

which can now be solved for y (apply formula or separation of variables).

Finally, substitute back $u = y^{1/(1-\alpha)}$.

[*]From the Swiss mathematician Jacob Bernoulli (1655-1705)

Example 2

Find the general solution of the ODE $u' = u + 2u^5$, for $u = u(t)$.

It is a Bernoulli equation with $a(t) = 1$, $b(t) = 2$ and $\alpha = 5$: we apply the substitution $y(t) = u^{1-\alpha}(t) = u^{-4}(t) \implies y'(t) = -4u'(t)u^{-5}(t)$.

Example 2

Find the general solution of the ODE $u' = u + 2u^5$, for $u = u(t)$.

It is a Bernoulli equation with $a(t) = 1$, $b(t) = 2$ and $\alpha = 5$: we apply the substitution $y(t) = u^{1-\alpha}(t) = u^{-4}(t) \implies y'(t) = -4u'(t)u^{-5}(t)$.

Rewrite the ODE as

$$\frac{u'}{u^5} = \frac{u}{u^5} + 2\frac{\cancel{u^5}}{\cancel{u^5}} \implies \frac{-y'}{4} = y + 2 \implies y' = -4(y+2) \to 1^{\text{st}} \text{ order, linear ODE}$$

Solving now in y returns: $y(t) = Ce^{-4t} - 2$, $C \in \mathbb{R}$.

The solution of the original ODE is thus:

$$u(t) = \pm y^{-1/4}(t) = \pm \frac{1}{(Ce^{-4t} - 2)^{1/4}}$$

Exercises

Exercise 1. Find the general solution of each of the following separable variables ODEs, then determine the respective solutions of the related problem under the constraint $u(1) = 1/2$.

(*i*) $u' = 4t^3\sqrt{t},\ \ t > 0$

(*ii*) $u' + \dfrac{2x}{u}(1 + 2x^2) = 0$

(*iii*) $u' = u^2 - 1$

(*iv*) $tu' = \sqrt{1 - u^2},\ t \in (1, 2)$

Exercise 2. Solve by substitution the following Bernoulli ODEs:

(*i*) $u' + tu - tu^3 = 0$

(*ii*) $t^2 u' - u^4 = tu,\ \ t > 2$

(*iii*) $x' - e^t\sqrt{x} = -2x,\ \ x > 0;$

Exercise 3. Determine a particular solution of each ODE employing the indicated Ansatz.

(*i*) $u' + 6u^2 = 1/t^2,\ t > 0$ with Ansatz $u(t) := \dfrac{\alpha}{t} + \beta$

(*ii*) $y' = 1 - t^2 + y^2$ with Ansatz a polynomial of degree 1 in t

(*iii*) $x^3 u' + x^2 u - u^2 = 2x^4,\ x > 1$ with Ansatz a polynomial of degree 2 in x

Exercise 4. For $0 < t < 1$, consider the differential equation in $u = u(t)$:

$$u' + 12tu^{2/3} + 3tu = 0\,. \tag{2}$$

(i) Determine the general solution of (2).

(ii) Find the solution $u(t)$ of the IVP satisfying equation (2) and $u(0) = u_0$ as a function depending on $u_0 \geq 0$.

(iii) Determine the point t_0 such that u vanishes in t_0.

(iv) Verify that any function

$$\tilde{u}(t) := \begin{cases} u(t)\,, & t \in [0, t_0] \\ 0\,, & t \in (t_0, 1) \end{cases} \tag{3}$$

with initial value $u_0 < 64(e^{1/2} - 1)^3$ is a solution of the IVP in (ii).

Appendix

Table of integrals

1. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \dfrac{1}{x}\,dx = \ln|x|$

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \dfrac{a^x}{\ln a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \csc^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \csc x \cot x\,dx = -\csc x$

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\int \tan x\,dx = \ln|\sec x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sinh x\,dx = \cosh x$

16. $\int \cosh x\,dx = \sinh x$

17. $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$

18. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right)$

*19. $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$

*20. $\int \dfrac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

AUDITORIUM EXERCISE CLASS 2

Exercise 1

(i) $\begin{cases} u' = 4t^3\sqrt{t}, \ t>0 \\ u(1) = \frac{1}{2} \end{cases}$ ← 1^{st} order ODE (linear)
sep. variables

$$u = \int du \underset{\uparrow}{=} \int u'(t)\,dt = \int 4t^{7/2}\,dt = \frac{8}{9}t^{9/2} + c \Rightarrow \left[u(t) = \frac{8}{9}t^{9/2} + c, \ c\in\mathbb{R}\right]$$
gen. sol. ODE

$u(t) \mapsto u$
$du = u'(t)\,dt$

Initial value:
$$\frac{1}{2} = u(1) = \frac{8}{9}\cdot 1 + c \Rightarrow c = -\frac{7}{18} \rightsquigarrow \left[u(t) = \frac{8}{9}t^{9/2} - \frac{7}{18}\right] \text{ sol. IVP}$$

(ii) $\begin{cases} u'(x) = -\frac{2x}{u}(1+2x^2), \ u\neq 0 \\ u(1) = \frac{1}{2} \end{cases}$ → 1^{st} order ODE, separable variables, $u = u(x)$

Rewrite: $u'\cdot u = -2x - 4x^3$

$$\frac{u^2}{2} = \int u\,du \underset{\uparrow}{=} \int u'\,u\,dx = -\int 2x\,dx - \int 4x^3\,dx = -x^2 - x^4 + c$$

$u = u(x)$
$du = u'(x)\,dx$

$$u^2 = 2c - 2x^2 - 2x^4 \Rightarrow u(x) = \pm\sqrt{2c - 2x^2 - 2x^4}, \ c\in\mathbb{R}$$

IVP: $\frac{1}{2} = u(1) \underset{\uparrow}{=} \sqrt{2c-2-2} = (2c-4)^{1/2} \Rightarrow 2c = \frac{17}{4} \longrightarrow u(x) = \sqrt{\frac{17}{4} - 2x^2 - 2x^4}$
positive $\sqrt{\ }$

(iii) $\begin{cases} t\cdot u' = \sqrt{1-u^2}, \ t\in(1,2) \\ u(1) = \frac{1}{2} \end{cases}$ $u = u(t)$

- If $u \equiv \pm 1 \to u' \equiv 0 \rightsquigarrow$ sol. ODE, but NOT of the IVP!
- Suppose $u \neq \pm 1 \to \frac{u'}{\sqrt{1-u^2}} = \frac{1}{t} \rightsquigarrow \arcsin(u) = \int \frac{du}{\sqrt{1-u^2}} = \int \frac{u'\,dt}{\sqrt{1-u^2}} = \int \frac{1}{t}\,dt = \ln(t) + c \Rightarrow$ ($u'\,dt = du$)

RECALL: $\left[\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right), \ |u|<a\right]$

$$\Rightarrow u(t) = \sin(\ln(t) + c), \ c\in\mathbb{R} \leftarrow \text{gen. sol.}$$

$$\frac{1}{2} = u(1) = \sin(\ln(1) + c) = \sin(c) \Rightarrow c = \frac{\pi}{6}$$

$$\left[u(t) = \sin\left(\ln(t) + \frac{\pi}{6}\right)\right] \text{ sol. IVP} \qquad \arcsin: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Exercise 2

Bernoulli: $\left[u'(t) = a(t)\cdot u(t) + b(t)\cdot u^{\alpha}(t)\right]$

(i) $u' + tu - tu^3 = 0$, $u = u(t)$

Let $y(t) := u^{1-\alpha}(t) = u^{1-3}(t) = \frac{1}{u^2(t)}$ ← for $u \neq 0$

$\hookrightarrow u' = \underbrace{-t}_{a(t)}u + \underbrace{t}_{b(t)}u^{3}$, $3 = \alpha$

- If $u \equiv 0$: solution ODE ✓
- If $u \neq 0$: $y = \frac{1}{u^2} \xrightarrow{\text{chain rule}} y' = -\frac{2}{u^3}u'$

$\frac{u'}{u^3} = -\frac{1}{2}y'$

$-\frac{1}{2}y' = \frac{u'}{u^3} = -\frac{tu}{u^3} + \frac{tu^3}{u^3} = -ty + t \Rightarrow -y' = -2ty + 2t$ ($\frac{u}{u^3} = \frac{1}{u^2}$)

(*) $\boxed{y' = 2ty - 2t}$ ← 1^{st} order, linear ODE in y

Solve (*) for y: $y' = 2t(y-1) \xrightarrow[\text{ex. by separation of variables}]{} \frac{y'}{y-1} = 2t \rightsquigarrow \int \frac{dy}{y-1} = \int \frac{y'}{y-1}\,dt = \int 2t\,dt = t^2 + c,\ c \in \mathbb{R} \Rightarrow$

$y \neq 1$; $\int \frac{dy}{y-1} = \ln|y-1|$

$\Rightarrow |y-1| = e^c \cdot e^{t^2} \Rightarrow y = \pm e^c \cdot e^{t^2} + 1$, $c \in \mathbb{R}$ $\rightsquigarrow$ gen. sol. is $\left[y(t) = c_1 e^{t^2} + 1,\ c_1 \in \mathbb{R}\right]$ of (*)

$\pm e^c = c_1 \neq 0$; can write $c_1 \in \mathbb{R}$ for $c_1 = 0$: $y \equiv 1$!

Substitute back:

$u^2(t) = \frac{1}{y(t)} = \frac{1}{c_1 e^{t^2} + 1} \Rightarrow \left[\begin{array}{l} u(t) = \pm\left(\frac{1}{c_1 e^{t^2}+1}\right)^{1/2},\ c_1 \in \mathbb{R} \\ \vee \\ u(t) \equiv 0 \end{array}\right]$ gen. sol. of Bernoulli ODE

Exercise 3

(ii) $y' = 1 - t^2 + y^2$, $y = y(t)$. Solve looking for sol. of the kind: $y(t) = at + b$, $a, b \in \mathbb{R}$ to be determined

Substitute: $y' = a \rightsquigarrow a = 1 - t^2 + (at+b)^2 = 1 - t^2 + a^2t^2 + 2abt + b^2 \Rightarrow a^2t^2 - t^2 + 2abt + 1 + b^2 - a = 0 \Rightarrow$

$\Rightarrow t^2\underbrace{(a^2-1)}_{\text{set } = 0} + \underbrace{2ab}_{=0} \cdot t + \underbrace{(1 + b^2 - a)}_{=0} = 0$, $\forall t \in I$

Use comparison of the coefficients: $\begin{cases} a^2 - 1 = 0 \Rightarrow a = \pm 1 \\ 2ab = 0 \Rightarrow \cancel{a=0} \vee b = 0 \\ 1 + b^2 - a = 0 \underset{b=0}{\rightsquigarrow} 1 - a = 0 \Rightarrow a = 1 \end{cases}$ $\rightsquigarrow$ We found $y(t) = 1\cdot t + 0 = t$ as a particular solution of the differential equation!

Exercise 4

$u'(t) + 12t \cdot u^{2/3}(t) + 3t\,u(t) = 0, \quad t \in (0,1)$

$\hookrightarrow u' = \underbrace{-12t}\, u^{2/3} \underbrace{-3t}\, u$ Bernoulli ODE with $\alpha = 2/3$

$a(t) = -3t$
$b(t) = -12t$

(i)

. $u \equiv 0$ solves ODE

. $u \neq 0$; $\quad y := u^{1-\alpha} = u^{1-\frac{2}{3}} = u^{1/3} = \sqrt[3]{u}$

$y' = \frac{1}{3u^{2/3}} u' \quad \leftarrow \quad y' = (1-\alpha)\left[a(t) \cdot y(t) + b(t)\right] = \frac{1}{3}\left[-3ty - 12t\right] = -ty - 4t$

Apply substitution: $\left[y' = -ty - 4t\right]$ 1st order, linear ODE in y

Solve: $A(t) = -t^2/2$
$B^*(t) = \int e^{t^2/2}(-4t)\,dt = -4e^{t^2/2} \rightsquigarrow y(t) = e^{A(t)}\left[B^*(t) + c\right] = e^{-t^2/2}\left[-4e^{t^2/2} + c\right] = c\,e^{-t^2/2} - 4, \quad c \in \mathbb{R}$ ← sol. ODE in y

Back to u: $u(t) = y^3(t) = \left(c e^{-t^2/2} - 4\right)^3, \; c \in \mathbb{R}$

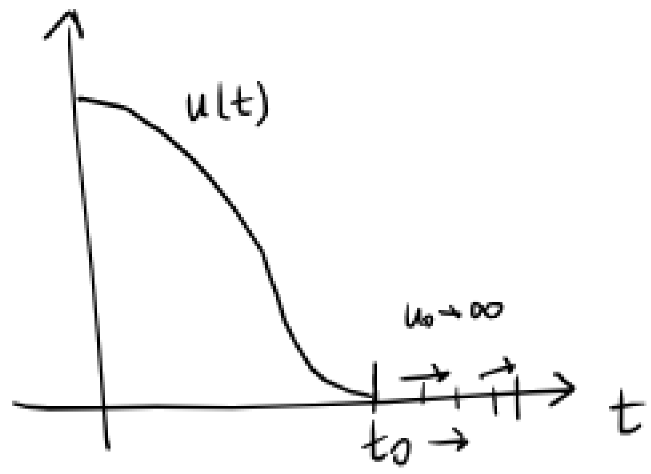
(ii) Plug $u(0) = u_0$: $\quad u_0 = u(0) = \left(c \cdot e^{0} - 4\right)^3 = (c-4)^3 \Rightarrow c = \sqrt[3]{u_0} + 4$

Solution IVP: $u(t) = \left(\left(4 + \sqrt[3]{u_0}\right) e^{-t^2/2} - 4\right)^3$

(iii) $\underset{\text{set}}{0} = u(t_0) = \left(\left(4 + \sqrt[3]{u_0}\right) e^{-t_0^2/2} - 4\right)^3 \Rightarrow \left(4 + \sqrt[3]{u_0}\right) e^{-t_0^2/2} = 4 \Rightarrow e^{-t_0^2/2} = \frac{4}{4 + \sqrt[3]{u_0}} \Rightarrow \ldots$

$\Rightarrow \underset{t_0 > 0}{t_0} = \left(\ln\left(\left(\underbrace{\frac{4 + \sqrt[3]{u_0}}{4}}_{\geq 1}\right)^2\right)\right)^{1/2}$

$t \mapsto \ln(t)$ and $t \mapsto \sqrt{t}$ are increasing functions:
the larger u_0, the larger t_0!



(iv) $\tilde{u}(t) = \begin{cases} u(t); & t \in [0, t_0] \\ 0; & t \in (t_0, 1) \end{cases}$ sol. ODE since $\tilde{u}\big|_{(0,t_0]} = u$ sol. ODE

$\tilde{u}\big|_{(t_0,1)} = 0$ sol. ODE

and for $u_0 < 64\left(e^{1/2} - 1\right)^3$, $t_0 < 1$