Auditorium Exercise Sheet 1

Differential Equations I for Students of Engineering Sciences

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General information on the DGL I course

- Lecturer: Professor Thomas Schmidt
- Lectures (English, weekly): Tue 16:45-18:15 Audimax II

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- Auditorium Exercise class (English, bi-weekly): Mo 09:45-11:15 H0.16
- Exercise groups (English, bi-weekly):
  - Mo 11:30-13:00 H0.01
  - Mo 16:00-17:30 N0009
  - Tue 08:00-09:30 O-007
  - Tue 15:00-16:30 D1.021

- Exercises/Homework at: DGL I - Lecture material WiSe 2024/25

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Introduction to ordinary differential equations

- Previously: given an algebraic equation/system, look for solution(s) among a certain vector space.

  Now: given a differential equation (ODE)/system of ODEs, look for solution(s) in a function space.

- A (real, scalar) ODE is an equation in which a function $u = u(t)$ and its derivative(s) $u', u'', \ldots, u^{(m)}$ (up to order $m \in \mathbb{N}$) are related:

$$F(t, u, u', u'', \ldots, u^{(m)}) = 0 \rightarrow m\text{-th order ODE in implicit form}$$

$$u^{(m)} = f(t, u, u', u'', \ldots, u^{(m-1)}) \rightarrow m\text{-th order ODE in explicit form}$$

  where $u: I \rightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ domain of definition.

- A function $\overline{u}$ defined on I is **solution** of $F(t, u, u', u'', \ldots, u^{(m)}) = 0$ if

$$F(t, \overline{u}(t), \overline{u}'(t), \overline{u}''(t), \ldots, \overline{u}^{(m)}(t)) = 0 \quad \text{for all } t \in I.$$

Introduction to ordinary differential equations

- Notice that in case an ODE admits a solution $\overline{u}$ on I, then we cannot expect $\overline{u}$ to be unique!
- The collection of all the possible solution of an ODE is called **general solution** of the ODE.

- Example 1: $u' = 4u$, where $u = u(t)$ defined on $I = \mathbb{R}$.

$$u \equiv 0 \text{ is a (trivial) solution}$$

$$u(t) = e^{4t} \text{ solves the ODE, but (for example) also } u(t) = 9e^{4t}!$$

The general solution of $u' = 4u$ is given by $u(t) = \mathbf{C}e^{4t},\ C \in \mathbb{R}$.

CHECK: $u'(t) = (Ce^{4t})' = 4Ce^{4t} = 4u(t)$ ✓

To determine a precise solution, we need to impose some restrictions to the ODE: initial and/or boundary conditions.

- Example of **initial value problem** (IVP):

$$\begin{cases} u'' = f(t,u,u') & \leftarrow \text{ ODE of order } m = 2 \\ u(t_0) = u_0 & \\ u'(t_0) = z_0 & \leftarrow \text{ need 2 conditions (here on } u \text{ and } u') \end{cases}$$

  with $t_0 \in I$, $u_0, z_0 \in \mathbb{R}$.

- Example of **boundary value problem** (BVP):

$$\begin{cases} u'' = f(t,u,u') & \leftarrow \text{ ODE of order } m = 2 \\ u(a) = u_a & \\ u(b) = u_b & \leftarrow \text{ need 2 boundary values}^* \end{cases}$$

  where $I = [a,b]$ and $u_a, u_b \in \mathbb{R}$.

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*In general, boundary condition of the form $r_i(u(a),u(b)) = 0$, for $i \in \{1,\dots,m\}$.

Example of initial value problem

- Solve the ODE in Example 1 with initial datum $u(2) = 5$, i.e. solve IVP given by the system

$$\begin{cases} u' = 4u \\ u(2) = 5. \end{cases}$$

Consider the general solution $u(t) = Ce^{4t},\ C \in \mathbb{R}$ of the ODE and find the appropriate C by substituting the initial condition:

$$5 = u(2) = Ce^{8} \implies C = 5e^{-8},$$

therefore the (unique!) solution of the IVP is given by $u(t) = 5e^{4(t-2)}$.

Linear differential equations

An m-th order ODE in $u = u(t)$ is **linear** if the ODE is a linear combination of the derivatives $u, u', \ldots, u^{(m)}$, i.e. it is of the form

$$A_m(t)u^{(m)} + A_{m-1}(t)u^{(m-1)} + \cdots + A_2(t)u'' + A_1(t)u' + A_0(t)u = b(t) \quad (1)$$

with coefficients given by the functions $A_0, A_1, \ldots, A_m \colon I \to \mathbb{R}$ and inhomogeneity term $b \colon I \to \mathbb{R}$.

Linear differential equations

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with coefficients given by the functions $A_0, A_1, \dots, A_m \colon I \to \mathbb{R}$ and inhomogeneity term $b \colon I \to \mathbb{R}$.

- If the coefficients $A_0, A_1, \dots, A_m$ are constant in I, we say that (1) is a linear ODE with **constant coefficients**.
- In case $b \equiv 0$ in I, the equation (1) is called **homogeneous**, otherwise it is **inhomogeneous**.
- For a linear ODE (1), the **corresponding homogeneous** ODE is the equation

$$A_m(t)u^{(m)} + A_{m-1}(t)u^{(m-1)} + \cdots + A_2(t)u'' + A_1(t)u' + A_0(t)u = 0 \quad (2)$$

Examples of linear/non-linear ODEs

- $u' = 4u$ (lin. hom. const. coeff.)
- $u' = 4u + 2e^t$ (lin. inhom. const. coeff.)
- $9u''/u - u + u^3 = 7$ (non-lin. inhom.) $\rightsquigarrow$ related hom: $9u''/u - u + u^3 = 0$
- $5\cos^4(t)u''' - u' + t^2 = 0$ (lin. inhom., non-const. coeff.) $\rightsquigarrow$ related hom: $5\cos^4(t)u''' - u' = 0$
- $u''' - 3u' = u^2$ (non lin. hom.)

NOTE: An m-th order **linear** ODE in $u = u(t)$ with coeff. $A_0, A_1, \dots, A_m, b \colon I \to \mathbb{R}$ is

$$A_m(t)u^{(m)} + A_{m-1}(t)u^{(m-1)} + \cdots + A_2(t)u'' + A_1(t)u' + A_0(t)u = b(t)$$

Resolution of first-order linear ODEs

A linear ODE of order $m = 1$ is of the kind

$$u' = a(t)u + b(t).$$

For a, b continuous, there is an explicit formula to find its general solution.

Resolution of first-order linear **HOMOGENEOUS** ODEs

Consider the differential equation

$$u' = a(t)u, \tag{3}$$

with $a\colon I \to \mathbb{R}$ continuous, let $A(t) := \int a(t)\,\mathrm{d}t$. The general solution of (3) is given by

$$u(t) = Ce^{A(t)}, \qquad C \in \mathbb{R}.$$

There are infinitely many solutions (since $C \in \mathbb{R}$)!

Resolution of first-order linear ODEs

A linear ODE of order $m = 1$ is of the kind

$$u' = a(t)u + b(t).$$

For a, b continuous, there is an explicit formula to find its general solution.

Resolution of first-order linear **INHOMOGENEOUS** ODEs

Consider the differential equation

$$u' = a(t)u + b(t), \tag{4}$$

with $a, b\colon I \to \mathbb{R}$ continuous, let $A(t) := \int a(t)\mathrm{d}t$, $B^*(t) := \int b(t)e^{-A(t)}\mathrm{d}t$.
The general solution of (3) is given by

$$u(t) = e^{A(t)}(B^*(t) + C), \qquad C \in \mathbb{R}.$$

There are infinitely many solutions (since $C \in \mathbb{R}$)!

Example. Resolution of a first-order linear homogeneous ODE

Example 2. Find the general solution $u = u(t)$, $t \in \mathbb{R}$ of

$$u' - 6t^2 u = 0. \tag{5}$$

It is a first order, linear ODE with $a(t) = 6t^2$ continuous and $b(t) \equiv 0$.
Employ the formula for homogeneous ODEs: $u(t) = Ce^{A(t)}$, $C \in \mathbb{R}$.
Compute $A(t) := \int a(t)\mathrm{d}t = \int 6t^2 = 2t^3$ (+const., set const. = 0).

The general solution of (5) is given by:

$$u(t) = Ce^{2t^3}, \qquad C \in \mathbb{R}.$$

If possible, CHECK that Ce^{2t^3} is solution!

Example. Resolution of a first-order linear inhomogeneous ODE

Example 3. Find the general solution $u = u(t)$, $t \in \mathbb{R}$ of

$$u' - 6t^2 u = t^2. \tag{6}$$

It is a first order, linear ODE with $a(t) = 6t^2$ and $b(t) = t^2$. Employ the formula for inhomogeneous ODEs: $u(t) = e^{A(t)}(B^*(t) + C)$, $C \in \mathbb{R}$.
Compute $A(t) := \int a(t)\mathrm{d}t = \int 6t^2 = 2t^3$ (+const., set const. = 0) and $B^*(t) := \int b(t)e^{-A(t)}\mathrm{d}t = \int t^2 e^{-2t^3}\mathrm{d}t = -e^{-2t^3}/6$.

The general solution of (6) is given by:

$$u(t) = e^{2t^3}(-e^{-2t^3}/6 + C) = Ce^{2t^3} - 1/6, \qquad C \in \mathbb{R}.$$

If possible, CHECK that $Ce^{2t^3} - 1/6$ is solution!

Example. Resolution of a first-order linear inhomogeneous ODE

Example 3. Find the general solution $u = u(t)$, $t \in \mathbb{R}$ of

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The general solution of (6) is given by:

$$u(t) = e^{2t^3}(-e^{-2t^3}/6 + C) = Ce^{2t^3} - 1/6, \qquad C \in \mathbb{R}.$$

NOTE: The general integral of (6) is given by the difference of $u_{\mathrm{hom}} := Ce^{2t^3}$ gen. int. of (5), and $u_{\mathrm{p}} := -1/6$ particular sol. of (6).

Exercises

Exercise 1. Determine the general solutions of the following (1st order, linear) ODEs. Unless specified, consider $I = \mathbb{R}$.

(i) $u' - 5u + 10 = t$

(ii) $\dot{x} = \cos(5t)(x+1)$

(iii) $u' + 2tu = e^{t-t^2}$

(iv) $y' + 4t^3y - \sin(t)e^{-t^4} = 0$

(v) $u'/2 - u = 2t^2$

(vi) $u' = \dfrac{1}{t^2-1}, -1 < t < 1$

(vii) $(y')^2 = y'\sin(3t)$

$(viii)$ $(y')^2 = y'\sin^2(3t)$

Exercise 2. For each ODE of Exercise 1, determine the (unique) function satisfying the ODE and having value -1 at time $t = 0$.

Exercises

Exercise 3. Consider the differential equation

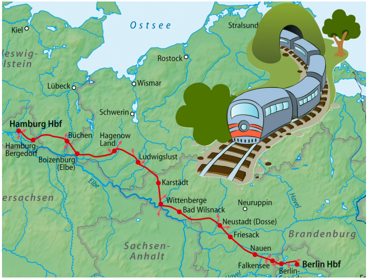
$$u'' + 9u - 3t = 0 \tag{7}$$

and the functions $u_1(t) := \cos(3t)$, $u_2(t) := \sin(3t)$, $u_3(t) := t/3$.

- Classify the equation (7).
- Verify that the functions $u_1 + u_3$, $u_2 + u_3$ and u_3 are solutions of (7).
- Write the associated homogeneous equation of (7).
- Verify that any linear combination of u_1, u_2 is a solution of the homogeneous equation of (7).
- Can you find a linear combination of u_1 and u_3 which does NOT solve (7)? Why?

Exercise 4.

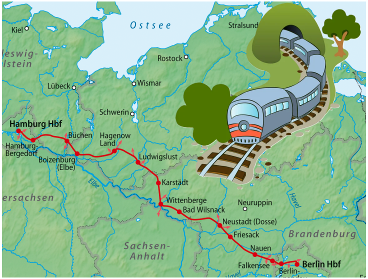
The ICE train Hamburg-Berlin travels a distance of 317 km departing at time $t_0 = 0s$ with a constant acceleration $a = 3m/40s^2$. The train should reach Berlin HBF after 3 hours and 10 minutes. Unfortunately, the train has to stop after just 10 minutes due to unexpected construction work...



Exercise 4.

The ICE train Hamburg-Berlin travels a distance of 317 km departing at time $t_0 = 0s$ with a constant acceleration $a = 3m/40s^2$. The train should reach Berlin HBF after 3 hours and 10 minutes. Unfortunately, the train has to stop after just 10 minutes due to unexpected construction work... After 2 hours, the train can continue on its route, and the driver tries to make up for the time lost by setting an acceleration (in km/h^2) proportional to the instantaneous speed (in km/h) by a constant 4, summed by 1.

- Does the ICE arrive at Berlin HBF with a delay of less than 1 hour?
- Sketch the graph of $u = u(t)$ representing the position at time $t \geq 0$.



Appendix

Table of integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \frac{1}{x}\,dx = \ln|x|$

3. $\int e^x\,dx = e^x$

4. $\int a^x\,dx = \frac{a^x}{\ln a}$

5. $\int \sin x\,dx = -\cos x$

6. $\int \cos x\,dx = \sin x$

7. $\int \sec^2 x\,dx = \tan x$

8. $\int \csc^2 x\,dx = -\cot x$

9. $\int \sec x \tan x\,dx = \sec x$

10. $\int \csc x \cot x\,dx = -\csc x$

11. $\int \sec x\,dx = \ln|\sec x + \tan x|$

12. $\int \csc x\,dx = \ln|\csc x - \cot x|$

13. $\int \tan x\,dx = \ln|\sec x|$

14. $\int \cot x\,dx = \ln|\sin x|$

15. $\int \sinh x\,dx = \cosh x$

16. $\int \cosh x\,dx = \sinh x$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

*19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Appendix

Some trigonometric identities

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta - \phi) - \cos(\theta + \phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta - \phi) + \cos(\theta + \phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta + \phi) - \sin(\theta - \phi)]}{2}$$

AUDITORIUM EXERCISE CLASS 1

EXERCISE 1+2

$u = u(t)$

$u' = a(t)\cdot u + b(t)$

$u(t) = e^{A(t)}(B^*(t)+C), \quad C\in\mathbb{R}$

$A(t) = \int a(t)dt$

$B^*(t) = \int e^{-A(t)} b(t) dt$

(i) $u' - 5u + 10 = t$ ← 1^{st}-order linear ODE inhom.

$u' = \underbrace{5}_{a(t)}u + \underbrace{(t-10)}_{b(t)}$

$A(t) = \int 5dt = 5t + \cancel{c_1} = 5t$

$B^*(t) = \int e^{-5t}(t-10)dt = -10\int e^{-5t}dt + \underline{\int te^{-5t}dt} = \frac{-10e^{-5t}}{-5} - \frac{t}{5}e^{-5t} + \frac{1}{5}\int e^{-5t}dt = \left(2-\frac{t}{5}\right)e^{-5t} + \frac{1}{5}\frac{e^{-5t}}{-5} =$

$\int fg' = fg - \int f'g$ (int. by parts)

$f(t) = t \quad f'(t) = 1$

$g'(t) = e^{-5t} \quad g(t) = \frac{e^{-5t}}{-5}$

polynomial · exponential (power 1 of t)

f g'

$= (50-5t-1)\frac{e^{-5t}}{25} = (49-5t)\frac{e^{-5t}}{25}$

Sol: $u(t) = e^{5t}\left(C + \frac{49-5t}{25}e^{-5t}\right) = \frac{49-5t}{25} + Ce^{5t}, \quad C\in\mathbb{R}$

Set IV: $u(0) = -1$

Solve: $-1 = u(0) = \frac{49}{25} + C\cdot\cancel{e^0} \Rightarrow C = -\frac{74}{25}$

Sol. IVP: $u(t) = \frac{49-5t-74e^{5t}}{25}$

(ii) $\dot{x}(t) = \cos(5t)\cdot(x(t)+1) =$

$= \underbrace{\cos(5t)}_{a(t)}\cdot x + \underbrace{\cos(5t)}_{b(t)}$

1^{st} order, linear ODE, inhom.

$A(t) = \int \cos(5t)dt = \frac{\sin(5t)}{5}$

$B^*(t) = \int e^{-\sin(5t)/5}\cdot\cos(5t)dt = -e^{-\sin(5t)/5}$

$x(t) = e^{\sin(5t)/5}\left(-e^{-\sin(5t)/5} + c\right) = C\cdot e^{\sin(5t)/5} - 1, \quad t\in\mathbb{R}, \; C\in\mathbb{R}$

Set IV: $-1 = x(0) = C\cdot e^{\sin(0)/5} - 1 = C - 1 \Rightarrow C = 0$ → sol. IVP ; $x(t) = -1, \quad t\in\mathbb{R}$

(iii) $u'(t) = \underbrace{\frac{1}{t^2-1}}_{b(t)}, \quad t\in(-1,1)$

1^{st} order, lin ODE

$a(t) = 0 \rightarrow A(t) = 0$

$B^*(t) = \int \cancel{e^0}\cdot\frac{1}{t^2-1}dt = \int\frac{1}{2(t-1)}dt - \int\frac{1}{2(t+1)}dt = \frac{1}{2}(\ln|t-1| - \ln|t+1|)$

$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{At+A+Bt-B}{(t^2-1)} = \frac{t(A+B)+(A-B)}{t^2-1}, \; \forall t \iff \begin{cases} A+B=0 \\ A-B=1\end{cases} \iff A = \frac{1}{2}, \; B = -\frac{1}{2}$

partial fraction dec.

comparison of the coefficients

$u(t) = e^0\left(C + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right) = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C, \quad C\in\mathbb{R}$

Set IV: $-1 = u(0) = \frac{1}{2}\cancel{\ln(1)} + C \Rightarrow C = -1$ ⇝ sol. IVP: $u(t) = -1 + \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$.

EXERCISE 3

(7) $u'' + 9u - \underline{3t} = 0$ 2nd order ODE, linear, const. coefficients, inhomogeneous

$u'' = 3t - 9u$

$u_1(t) = \cos(3t)$
$u_2(t) = \sin(3t)$
$u_3(t) = t/3$

- Is $u_1 + u_3$ sol of (7)? $(u_1+u_3)' = -3\sin(3t) + \frac{1}{3}$
  $(u_1+u_3)'' = -9\cos(3t) \underset{??}{=} 3t - 9(u_1+u_3) = 3t - 9\left(\cos(3t) + \frac{t}{3}\right)$ ✓ YES

- Is $u_2 + u_3$ sol. of (7)? $(u_2+u_3)' = 3\cos(3t) + \frac{1}{3}$
  $(u_2+u_3)'' = -9\sin(3t) \underset{??}{=} 3t - 9(u_2+u_3) = 3t - 9\left(\sin(3t) + \frac{t}{3}\right)$ ✓ YES

- Is u_3 sol of (7)? $u_3' = \frac{1}{3}$, $u_3'' = 0$
  $0 = u_3'' \underset{??}{=} 3t - 9u_3 = 3t - 9 \cdot \frac{1}{3}t$ ✓ YES

Corresponding homogeneous ODE: $\boxed{u'' + 9u = 0}$ (hom.)

Linear combination of u_1, u_2 is: $\tilde{u}(t) = \alpha \cdot u_1(t) + \beta \cdot u_2(t)$, $\alpha, \beta \in \mathbb{R}$

$\tilde{u}' = \alpha u_1' + \beta u_2'$
$\tilde{u}'' = \alpha u_1'' + \beta u_2''$

$$\tilde{u}'' + 9\tilde{u} = \alpha u_1'' + \beta u_2'' + 9\alpha u_1 + 9\beta u_2 =$$
$$= \alpha(u_1'' + 9u_1) + \beta(u_2'' + 9u_2) =$$
$$= \alpha\underbrace{\left(\cancel{-9\cos(3t) + 9\cos(3t)}\right)}_{0} + \beta\underbrace{\left(\cancel{-9\sin(3t) + 9\sin(3t)}\right)}_{0} = 0, \ \forall \alpha, \beta \in \mathbb{R}$$

Meaning: any linear combination of u_1 and u_2 solves (hom)

Linear combination of u_1, u_3 is: $\tilde{\tilde{u}} := \alpha u_1 + \beta u_3$, $\alpha, \beta \in \mathbb{R}$

$$\tilde{\tilde{u}}'' + 9\tilde{\tilde{u}} - 3t = \alpha u_1'' + \beta u_3'' + 9\alpha u_1 + 9\beta u_3 - 3t = \underbrace{\cancel{\alpha(u_1'' + 9u_1)}}_{0} + \beta(u_3'' + 9u_3) - 3t = \beta \cdot 3t - 3t = 3t(\beta - 1)$$

In order for $\tilde{\tilde{u}}$ to solve (7) we need: $\tilde{\tilde{u}}'' + 9\tilde{\tilde{u}} - 3t = 0, \forall t$, namely $3t(\beta-1) = 0$, $\forall t \in \mathbb{R} \Leftrightarrow \beta = 1$

Only combinations of the type $\boxed{\alpha u_1 + u_3, \ \alpha \in \mathbb{R}}$ can solve (7).