Differential Equations I

Exam WiSe 24/25 – Solution

Problem 1. (5 points)

(a) Solve the following initial value problem:

$$u'(t) = -(2t+1)u(t) + \mathrm{e}^{-t^2} \quad \text{for } \ t \geq 0, \qquad u(0) = 1.$$

(b) Solve the following initial value problem by applying a suitable substitution:

$$u'(t) = (2t+1)u(t) - \mathrm{e}^{-t^2}(u(t))^2 \quad \text{for } \ t \geq 0, \qquad u(0) = 1.$$

Solution.

(a) We have a linear equation of the form $u'(t) = a(t)u(t) + b(t)$ with $a(t) = -(2t+1)$ and $b(t) = \mathrm{e}^{-t^2}$. We choose

$$A(t) = -(t^2 + t)$$

as a primitive of a and and get the solution of the initial value problem by applying the solution formula with $t_0 = 0$ and $y_0 = 1$:

$$\begin{aligned} u(t) &= \mathrm{e}^{A(t)} \cdot \left[\int_{t_0}^t \mathrm{e}^{-A(s)} b(s) \, \mathrm{d}s + y_0 \mathrm{e}^{-A(t_0)}\right] = \mathrm{e}^{-(t^2+t)} \cdot \left[\int_0^t \mathrm{e}^{(s^2+s)} \mathrm{e}^{-s^2} \, \mathrm{d}s + 1\right] \\ &= \mathrm{e}^{-(t^2+t)} \cdot \left[\int_0^t \mathrm{e}^{s} \, \mathrm{d}s + 1\right] = \mathrm{e}^{-(t^2+t)} \left[\mathrm{e}^t - 1 + 1\right] = \mathrm{e}^{-t^2}. \end{aligned}$$

(b) We have a *Bernoulli* differential equation with

$$a(t) = (2t+1), \qquad b(t) = -\mathrm{e}^{-t^2}, \qquad \alpha = 2.$$

The substitution

$$y(t) = u(t)^{1-\alpha}, \qquad y(0) = u(0)^{1-\alpha}$$

leads to

$$y'(t) = (1-\alpha)a(t)y(t) + (1-\alpha)b(t) = -(2t+1)y(t) + \mathrm{e}^{-t^2}, \qquad y(0) = 1.$$

From part (a) we know this has the solution $y(t) = \mathrm{e}^{-t^2}$. We find the solution u from the back-substitution

$$u(t) = y(t)^{\frac{1}{1-\alpha}} = y(t)^{-1} = \mathrm{e}^{t^2}.$$

Problem 2. (5 points)

Consider the differential equation

$$u'''(t) + 2u''(t) = 0 \qquad \text{for } t \geq 0. \tag{$*$}$$

(a) Determine the general solution of this equation.

(b) Is the initial condition $u(0) = 1$ enough to characterize a unique solution of $(*)$? Explain your answer.

(c) For which $a, b \in \mathbb{R}$ does the solution u of the initial value problem for $(*)$ with

$$u(0) = 1, \quad u'(0) = a, \quad u''(0) = b$$

satisfy the condition

$$\lim_{t\to\infty} u(t) = 0?$$

Solution.

(a) The characteristic polynomial is

$$p(\lambda) = \lambda^3 + 2\lambda^2 = \lambda^2(\lambda + 2)$$

with roots

$$\lambda_{1,2} = 0, \qquad \lambda_3 = -2.$$

With that we get the fundamental system

$$\left\{w_1(t) = \mathrm{e}^{-2t},\ w_2(t) = \mathrm{e}^{0\cdot t} = 1,\ w_3(t) = t\mathrm{e}^{0\cdot t} = t\right\}$$

and the general solution is

$$u(t) = c_1\mathrm{e}^{-2t} + c_2 + c_3 t, \qquad c_1, c_2, c_3 \in \mathbb{R}.$$

(b) No, this is not enough to single out a unique solution. E.g. both $u_1(t) = 1$ and $u_2(t) = \mathrm{e}^{-2t}$ solve this initial value problem.

(c) Every solution u has the form $u(t) = c_1\mathrm{e}^{-2t} + c_2 + c_3 t$. In order to have $\lim_{t\to\infty} u(t) = 0$, we need $c_2 = c_3 = 0$. From $u(0) = 1$ it follows that $c_1 = 1$.

That means $u(t) = \mathrm{e}^{-2t}$ is the olny fuction that solves the differential equation and satisfies both $u(0) = 1$ and $\lim_{t\to\infty} u(t) = 0$. We have

$$u'(t) = -2\mathrm{e}^{-2t} \quad \Rightarrow \quad u'(0) = -2,$$
$$u''(t) = 4\mathrm{e}^{-2t} \quad \Rightarrow \quad u''(0) = 4.$$

This means $a = -2,\ b = 4$.

Problem 3. (4 points)

Consider the differential equation

$$(t - \sin(t)\cos(t))u^2 \; + \; (t^2 + \cos^2(t) + 1)u \cdot u' = 0 \qquad \text{for} \;\; t \geq 0.$$

(a) Show that this is an exact differential equation.

(b) Determine a potential for this equation.

(c) Solve the corresponding initial value problem with $u(0) = 1$.

Solution.

(a) With

$$f(t,u) = (t - \sin(t)\cos(t))u^2, \qquad g(t,u) = (t^2 + \cos^2(t) + 1)u$$

the equation has the form

$$f(t,u) + g(t,u)u' = 0.$$

It holds that

$$\frac{\partial f}{\partial u}(t,u) \;=\; 2(t - \sin(t)\cos(t))u \;=\; \frac{\partial g}{\partial t}(t,u),$$

and thus the equation is exact.

(b) We have

$$\int (t - \sin(t)\cos(t))u^2 \; \mathrm{d}t \;=\; \frac{1}{2}(t^2 + \cos^2(t))u^2 + K(u),$$
$$\int (t^2 + \cos^2(t) + 1)u \; \mathrm{d}u \;=\; \frac{1}{2}(t^2 + \cos^2(t) + 1)u^2 + M(t),$$

and so with

$$K(u) = \frac{1}{2}u^2, \qquad M(t) = 0$$

we get equality. A potential is

$$\Psi(t,u) \;=\; \frac{1}{2}(t^2 + \cos^2(t) + 1)u^2.$$

(c) For $C > 0$, the solution are given by

$$\Psi(t,u) = C \quad \Rightarrow \quad \frac{1}{2}(t^2 + \cos^2(t) + 1)u^2 = C \quad \Rightarrow \quad u = \pm\sqrt{\frac{2C}{t^2 + \cos^2(t) + 1}}.$$

It follows that

$$1 = u(0) = \sqrt{\frac{2C}{2}} \quad \Rightarrow \quad C = 1.$$

With this, the solution is

$$u(t) = \sqrt{\frac{2}{t^2 + \cos^2(t) + 1}}.$$

Problem 4. (6 points)

Consider the system of differential equations $u' = Au$ with

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

(a) Determine a fundamental system for this equation.

(b) Determine all equilibria of the system and check them for stability.

Solution.

(a) The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} -3-\lambda & 0 & 0 \\ 1 & -1-\lambda & 2 \\ 2 & 0 & 1-\lambda \end{pmatrix} = (-3-\lambda)(-1-\lambda)(1-\lambda)$$

with eigenvalues

$$\lambda_1 = -3, \qquad \lambda_2 = -1, \qquad \lambda_3 = 1.$$

Corresponding eigenvectors:

For $\lambda_1 = -3$:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right) \Rightarrow v^{[1]} = c \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \qquad c \in \mathbb{R}.$$

For $\lambda_2 = -1$:

$$\left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \Rightarrow v^{[2]} = c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad c \in \mathbb{R}.$$

For $\lambda_3 = 1$:

$$\left(\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v^{[3]} = c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad c \in \mathbb{R}.$$

A fundamental system is therefore

$$\left\{ \mathrm{e}^{-3t} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \quad \mathrm{e}^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{e}^{t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) All eigenvalues are non-zero. Therefore, $u^* = (0,0,0)^\top$ is the only equilibrium. Since $\lambda_3 > 0$, it is unstable.