Differential Equations I

Exam SoSe 2025 - Solutions

Problem 1. (5 points)

For each of the following differential equations, find the general solution by using the method of separation of variables.

After that, solve the corresponding initial value problems with the given initial data and determine for which t the respective solutions exist.

(a) $2t^2u' = u^2$ for $t \geq 1$, $u(1) = 4$,

(b) $u'u = \mathrm{e}^t$ for $t \geq 0$, $u(0) = 2$.

Solution.

(a) The equation has the form $u' = u^2 \cdot \frac{1}{2t^2}$. One solution is $u = 0$. For $u \neq 0$ we get the general solution:

$$\int \frac{1}{u^2}\,\mathrm{d}u = \int \frac{1}{2t^2}\,\mathrm{d}t \quad \Rightarrow \quad -\frac{1}{u} = -\frac{1}{2t} + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad u(t) = \frac{2t}{1-2ct}.$$

From the initial data we find

$$u(1) = \frac{2}{1-2c} \overset{!}{=} 4 \quad \Rightarrow \quad 2 = 4 - 8c \quad \Rightarrow \quad c = \frac{1}{4}.$$

With this we have

$$u(t) = \frac{2t}{1-t/2}$$

and the solution is defined for $t \in [1, 2)$.

(b) The equation has the form $u' = \frac{1}{u} \cdot \mathrm{e}^t$. The function $g(u) = 1/u$ has no zeros. We get the general solution:

$$\int u\mathrm{d}u = \int \mathrm{e}^t\,\mathrm{d}t \quad \Rightarrow \quad \frac{1}{2}u^2 = \mathrm{e}^t + c, \quad c \in \mathbb{R} \quad \Rightarrow \quad u = \pm\sqrt{2\mathrm{e}^t + 2c}.$$

From the initial data we find

$$u(0) = \pm\sqrt{2+2c} \overset{!}{=} 2 \quad \Rightarrow \quad 2 + 2c = 4 \quad \Rightarrow \quad c = 1$$

With this we have

$$u(t) = \sqrt{2\mathrm{e}^t + 2}$$

and sind $2\mathrm{e}^t + 2 > 0$ for all $t \geq$, the solution is defined for all $t \geq 0$.

Problem 2. (6 points)

Consider the matrix

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}.$$

(a) Determine a fundamental system for the solution of $u' = Au$.

(b) Determine all equilibria of $u' = Au$ and check them for stability.

Solution.

(a) The characteristic polynomial is

$$p(\lambda) = \det \begin{pmatrix} -1-\lambda & 2 \\ 2 & -4-\lambda \end{pmatrix} = (-1-\lambda)(-4-\lambda) - 4 = \lambda^2 + 5\lambda = \lambda(\lambda+5)$$

with eigenvalues.

$$\lambda_1 = -5, \qquad \lambda_2 = 0.$$

Corresponding eigenvectors:

For $\lambda_1 = -5$:

$$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v^{[1]} = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad c \in \mathbb{R}.$$

For $\lambda_2 = 0$:

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v^{[2]} = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad c \in \mathbb{R}.$$

With that a fundamental system is given by

$$\left\{ e^{-5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

(b) Since $\lambda_2 = 0$ is a simple eigenvalue, all solutions of $Au^* = 0$ are multiples of the corresponding eigenvector. That is,

$$u^* = c\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is an equilibrium for each $c \in \mathbb{R}$. Since $\lambda_1 < 0$ and $\lambda_2 = 0$ is simple, all equilibria are stable (but not asymptotically stable).

Problem 3. (4 points)

Let $A \in \mathbb{R}^{4\times 4}$ be a real matrix with eigenvalues

$$\lambda_1 = -2, \quad \lambda_2 = -1, \quad \lambda_3 = 2\mathrm{i}, \quad \lambda_4 = -2\mathrm{i}.$$

Which of the following statements are true, which are false? Explain your answers.

(1) The system $u' = Au$ has exactly one equilibrium.

(2) The equilibrium $u^* = (0,0,0,0)^\top \in \mathbb{R}^4$ is asymptotically stable.

(3) There exists a π-periodic solution of $u' = Au$.

(4) There exists a solution u of $u' = Au$ with $\lim_{t\to\infty} |u(t)| = \infty$.

Hint: In each case a short explanation is enough. You do not need long calculations.

Solution.

(1) The statement is *true*. All eigenvalues are non-zero. Therefore A is regular and $Au^* = 0$ has the unique solution $u^* = 0$.

(2) The statement is *false*. We have $\mathrm{Re}(\lambda_3) = \mathrm{Re}(\lambda_4) = 0$ and λ_2, λ_3 are simple eigenvalues. Then u^* is stable, but not asymptotically stable.

(3) The statement is *true*. Sice $\lambda_3 = 2\mathrm{i}, \lambda_4 = -2\mathrm{i}$ are purely imaginary eigenvalues, there exists a solution of the form

$$u(t) = \cos(2t)a \; + \; \sin(2t)b \qquad \text{mit} \;\; a, b \in \mathbb{R}^4.$$

(4) The statement is *false*. Let $v^{[k]}$ be the eigenvectors corresponding to $\lambda_k,\ 1 \le k \le 4$. Each solution has the form

$$u(t) = c_1 \mathrm{e}^{-2t} v^{[1]} \; + \; c_2 \mathrm{e}^{-t} v^{[2]} \; + \; c_3 \mathrm{e}^{2\mathrm{i}t} v^{[3]} \; + \; c_4 \mathrm{e}^{-2it} v^{[4]}$$

and therefore

$$|u(t)| \le \mathrm{e}^{-2t} |c_1 v^{[1]}| \; + \; \mathrm{e}^{-t} |c_2 v^{[2]}| \; + \; |c_3 \mathrm{e}^{2\mathrm{i}t} v^{[3]}| \; + \; |c_4 \mathrm{e}^{-2it} v^{[4]}|$$

The first to terms on the right-hand side go to zero as $t \to \infty$, and the last two are bounded.

Problem 4. (5 points)

Consider the differential equation

$$u''(t) - 2u'(t) + 2u(t) \;=\; 2\sin(t) - 4\cos(t).$$

(a) Determine a *real* fundamental system for the homogenous problem.

(b) Find a particular solution u_p of the inhomogeneous problem.

Hint: You can use the ansatz $u_p(t) = a\cos(t) + b\sin(t)$ with $a, b \in \mathbb{R}$.

Solution.

(a) The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 2\lambda + 2.$$

with roots

$$\lambda_{1,2} \;=\; 1 \pm \sqrt{1-2} \;=\; 1 \pm \mathrm{i}.$$

We get the complex fundamental system

$$\{\mathrm{e}^{(1+\mathrm{i})t},\ \mathrm{e}^{(1-\mathrm{i})t}\},$$

and a real fundamental system is $\{\mathrm{Re}(\mathrm{e}^{(1+\mathrm{i})t}),\ \mathrm{Im}(\mathrm{e}^{(1+\mathrm{i})t})\}$.
Using the Euler formula we get

$$\mathrm{e}^{(1+\mathrm{i})t} \;=\; \mathrm{e}^t \cdot \mathrm{e}^{\mathrm{i}t} \;=\; \mathrm{e}^t(\cos(t) + \mathrm{i}\sin(t)) \;=\; \mathrm{e}^t\cos(t) \;+\; \mathrm{i}\,\mathrm{e}^t\sin(t).$$

Then a real fundamental system is

$$\{\mathrm{e}^t\cos(t),\ \mathrm{e}^t\sin(t)\}.$$

(b) Using the ansatz $u_p(t) = a\cos(t) + b\sin(t)$ we get

$$\begin{aligned}
&u_p''(t) - 2u_p'(t) + 2u_p(t) \\
&= (-a\cos(t) - b\sin(t)) \;-\; 2(-a\sin(t) + b\cos(t)) \;+\; 2(a\cos(t) + b\sin(t)) \\
&= (a-2b)\cos(t) \;+\; (2a+b)\sin(t) \;\overset{!}{=}\; 2\sin(t) - 4\cos(t).
\end{aligned}$$

It follows that

$$a - 2b = -4, \qquad 2a + b = 2.$$

Adding twice the second equation to the first equation yields $a = 0$ and therefore $b = 2$. Finally,

$$u_p(t) = 2\sin(t).$$