Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I
for Students of Engineering Sciences
Sheet 7, Homework

Exercise 1: Consider the initial value problem be given.

$$u''(t) - u'(t) - 2u(t) = e^{2t} \cdot \sin(t), \qquad u(0) = u'(0) = 0.$$

a) Solve the initial value problem with the help of the Laplace transformation.

b) Determine the solution of the initial value problem without Laplace transformation. Proceed as follows.

(i) Determine the general solution of the corresponding homogeneous differential equation with help of the characteristic polynomial.

(ii) Rewrite the differential equation as a first order system and provide a fundamental matrix for this system.

(iii) Determine the general solution of the inhomogeneous differential equation. Use the method of variation of constants for the corresponding system.

(iv) Adjust the coefficients to the initial conditions.

Hint:

- $\displaystyle\int e^{\alpha t} \cdot \sin(t)\, dt = \frac{e^{\alpha t}}{\alpha^2 + 1} \left(\alpha \cdot \sin(t) - \cos(t)\right) + C\,.$
- Use Part b) as recap and a combination of techniques from the previous sheets!

Solution:

a) Transformation of the IVP

$u(t) \circ\!\!-\!\!\bullet\, U(s), \qquad u'(t) \circ\!\!-\!\!\bullet\, sU(s) - u(0) = sU(s),$

$u''(t) \circ\!\!-\!\!\bullet\, s^2U(s) - u'(0) = s^2U(s),$

$e^{2t} \cdot \sin(t) \circ\!\!-\!\!\bullet\, \dfrac{1}{(s-2)^2 + 1^2},$

The IVP becomes

$$(s^2 - s - 2)U(s) = \frac{1}{(s-2)^2 + 1} \iff (s-2)(s+1)U(s) = \frac{1}{s^2 + -4s + 5}.$$

with the solution

$$U(s) = \frac{1}{(s-2)(s+1)(s^2 - 4s + 5)}$$

Partial fraction decomposition ansatz:

$$U(s) = \frac{cs + d}{s^2 - 4s + 5} + \frac{b}{s-2} + \frac{a}{s+1}$$

returns the condition

$(cs + d)(s-2)(s+1) + b(s^2 - 4s + 5)(s+1) + a(s^2 - 4s + 5)(s-2) = 1\,.$

For $s = 2\, :\, b(1)(3) = 1 \implies b = \frac{1}{3}\,.$

For $s = -1 \,:\, a(10)(-3) = 1 \implies a = -\frac{1}{30}\,.$
For $s = 0 \,:\, d(-2)(1) + 5b - 10a = 1 \implies d = \frac{1}{2}\,.$
Comparison of coefficients for the power s^3 finally delivers
$c + b + a = 0 \implies c = -\frac{3}{10}\,.$
Thus it holds

$$\begin{aligned}
U(s) &= \frac{-\frac{3}{10}s + \frac{1}{2}}{(s-2)^2+1} + \frac{1}{3}\,\frac{1}{s-2} + -\frac{1}{30}\,\frac{1}{s+1} \\
&= \frac{-\frac{3}{10}(s-2) - \frac{6}{10} + \frac{1}{2}}{(s-2)^2+1} + \frac{1}{3}\,\frac{1}{s-2} + -\frac{1}{30}\,\frac{1}{s+1} \\
&= -\frac{3}{10}\,\frac{(s-2)}{(s-2)^2+1} - \frac{1}{10}\,\frac{1}{(s-2)^2+1} + \frac{1}{3}\,\frac{1}{s-2} - \frac{1}{30}\,\frac{1}{s+1} \\
&\bullet\!\!-\!\!\circ\; -\frac{3}{10}\,e^{2t}\cos(t) - \frac{1}{10}e^{2t}\sin(t) + \frac{1}{3}\,e^{2t} - \frac{1}{30}\,e^{-t} = u(t)\,.
\end{aligned}$$

b) (i) Characteristisch polynomial: $P(\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)\,.$
The zeroes of P are: $\lambda_1 = -1,\ \lambda_2 = 2\,.$
General solution: $u_h(t) = c_1 \cdot u^{[1]}(t) + c_2(t) \cdot u^{[2]}(t) = c_1 e^{-t} + c_2\, e^{2t}\,.$

(ii) With

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} u \\ u' \end{pmatrix}, \qquad \boldsymbol{u}' = \begin{pmatrix} u' \\ u'' \end{pmatrix}$$

one gets as equivalent system

$$\boldsymbol{u}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^{2t} \cdot \sin(t) \end{pmatrix}.$$

As fundamenal matrix one can choose

$$\boldsymbol{U}(t) = \begin{pmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{pmatrix}.$$

(iii) Ansatz: $u_p(t) = c_1(t) \cdot u^{[1]}(t) + c_2(t) \cdot u^{[2]}(t) = c_1(t) \cdot e^{-t} + c_2(t) \cdot e^{2t}\,.$
Inserting into the differential equation returns, with $b(t) =$ inhomogeneous term of the scalar differential equation

$$\boldsymbol{U}(t)\,\mathbf{c}'(t) = \begin{pmatrix} 0 \\ b(t) \end{pmatrix} \iff \begin{pmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{2t} \cdot \sin(t). \end{pmatrix}$$

Summing the two rows returns

$$3e^{2t}\,c_2' = e^{2t} \cdot \sin(t) \implies c_2' = \frac{1}{3}\sin(t)\,.$$

Thus we could choose $\boxed{c_2(t) = -\frac{1}{3}\cos(t)\,.}$
From the first row of the system we get

$$e^{-t}c_1' + c_2' e^{2t} = e^{-t}c_1' + \frac{1}{3}\sin(t)\,e^{2t} = 0\,.$$

Thus

$$c_1'(t) = -\frac{1}{3}\sin(t)\,e^{3t} \implies c_1(t) = -\int \frac{1}{3}\sin(t)\,e^{3t}\,dt\,.$$

With the help of the hint it follows

$$c_1(t) = -\frac{1}{30}\, e^{3t}\, \left(3\sin(t) - \cos(t)\right) + C\,.$$

A particular solution is for example

$$u_p(t) = -\frac{1}{30}\, e^{3t}\, \left(3\sin(t) - \cos(t)\right)\, e^{-t} - \frac{1}{3}\, \cos(t) e^{2t} = -\frac{e^{2t}}{10}\, \left(\sin(t) + 3\cos(t)\right)\,.$$

The general solution is therefore

$$u(t) = u_h(t) + u_p(t) = c_1 e^{-t} + c_2\, e^{2t} - \frac{e^{2t}}{10}\, \left(\sin(t) + 3\cos(t)\right).$$

(iv) The first initial condition $u(0) = 0$ returns:
$c_1 + c_2 - \dfrac{1}{10}\,(3) = 0\,.$
Since
$u'(t) = -c_1 e^{-t} + 2c_2\, e^{2t} - \dfrac{e^{2t}}{10}\, \left(2\sin(t) + 6\cos(t) + \cos(t) - 3\sin(t)\right)$
the second initial condition $u'(0) = 0$ returns
$u'(0) = -c_1 + 2c_2 - \frac{1}{10}\,(6+1) = 0\,.$
Sum of the two conditions returns:
$3c_2 - 1 = 0 \implies c_2 = \frac{1}{3}.$
For example, from the second condition we obtain
$c_1 = 2c_2 - \frac{7}{10} = -\frac{1}{30}$
and from this (as in Part a))

$$u(t) = -\frac{1}{30} e^{-t} + \frac{1}{3}\, e^{2t} - \frac{1}{10} e^{2t} \sin(t) - \frac{3}{10}\, e^{2t} \cos(t).$$

Exercise 2:

Consider the linear system $\boldsymbol{u}'(t) = \begin{pmatrix} -2 & 1 & -2\alpha \\ 0 & -1+\alpha & 0 \\ -2\alpha & -1 & -2 \end{pmatrix} \boldsymbol{u}(t)$.

Analyze the stability behavior of the stationary point $(0,0,0)^T$ depending on the parameter $\alpha \in \mathbb{R}$.

Solution:

Expansion with respect to the second row returns the characteristic polynomial:

$P(\lambda) = (-1+\alpha-\lambda)[(2+\lambda)^2 - 4\alpha^2]$.

Eigenvalues: $\lambda_1 = -1+\alpha, \quad \lambda_2 = -2+2\alpha, \quad \lambda_3 = -2-2\alpha$

$\alpha > +1 \iff \lambda_1 > 0 \iff \lambda_2 = 2\lambda_1 > 0$.

$\alpha < -1 \iff \lambda_3 > 0$.

$-1 < \alpha < 1$: negative real part of all eigenvalues. The zero solution is asymptotically stable.

For $\alpha < -1$ or $\alpha > +1$ there is at least one eigenvalue with positive real part. The zero solution is unstable.

$\alpha = -1 \implies \lambda_1 = -2, \lambda_2 = -4, \lambda_3 = 0$: The zero solution is stable.

$\alpha = +1 \implies \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -4$.

Eigenspace to the double eigenvalue zero:

$$\begin{pmatrix} -2 & 1 & -2 \\ 0 & 0 & 0 \\ -2 & -1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff v_2 = 0,\ v_3 = -v_1$$

The eigenspace of the eigenvalue zero with multiplicity two has dimension one. The zero solution is unstable.

Exercise 3:

In a two-population model (predator-prey model) $x(t)$ denotes the population of the prey species, $y(t)$ the population of the predator species at time t. The temporal growth of the populations is described by the following system of differential equations

$$\begin{aligned} x' &= x\,(x - 1 - y) \\ y' &= y\,(2x - 3 - y). \end{aligned}$$

a) Find all equilibrium points of the system.

b) Analyze the stability of every equilibrium point.

Solution:

a) For the equilibrium points it must hold

$$\begin{aligned} x' &= x\,(x - 1 - y) = 0 \\ y' &= y\,(2x - 3 - y) = 0. \end{aligned}$$

Thus $x = 0$ or $y = x - 1$
and $y = 0$ or $y = 2x - 3$.

We obtain the following points
$P_1 = (0,0)^T$
$x = 0$ and $y = 2x - 3$, hence $P_2 = (0,-3)^T$
$y = 0$ and $y = x - 1$, hence $P_3 = (1,0)^T$
$y = x - 1$ and $y = 2x - 3 \Longrightarrow x - 1 = 2x - 3 \Longleftrightarrow x = 2,\, y = 1$.
Hence $P_4 = (2,1)^T$.

b) Since the above system is non-linear, for the stability analysis we must consider the linearization of the right-hand side

$$F(x,y) = \begin{pmatrix} x\,(x - 1 - y) \\ y\,(2x - 3 - y). \end{pmatrix} = \begin{pmatrix} x^2 - x - xy \\ 2xy - 3y - y^2. \end{pmatrix}$$

in the points P_1 to P_4. It is

$$JF(x,y) = \begin{pmatrix} 2x - 1 - y & -x \\ 2y & -3 + 2x - 2y \end{pmatrix}$$

and from this

$JF(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$ with the eigenvalues -1 and -3.

P_1 is asymptotically stable.

$JF(0,-3) = \begin{pmatrix} +2 & 0 \\ -6 & 3 \end{pmatrix}$ with the eigenvalues 2 and 3.

P_2 is unstable.

$JF(1,0) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ with the eigenvalues 1 and -1.

P_3 is unstable.

$JF(2,1) = \begin{pmatrix} 2 & -2 \\ 2 & -1 \end{pmatrix} \Longrightarrow P(\lambda) = (2 - \lambda)(-1 - \lambda) + 4 = 0!$

$\lambda^2 - \lambda + 2 = 0 \Longrightarrow \lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{-7}{4}}$.

Since the real parts of both eigenvalues are positive, P_4 is unstable as well.

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