Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 7, Exercise class

To shorten the notation we employ the Doetsch symbol:

$$F(s) := \mathscr{L}f(s) := \int_0^\infty e^{-st} f(t)dt \iff f(t) \circ\!\!-\!\!\bullet\, F(s)$$

The following correspondences or relations for $\mathrm{Re}\,(s) > \gamma$, which were either proved in the lecture or could be proved completely analogously to the lectures procedure, may be used. We always have $f(t) = 0, \forall t < 0$.

$f(t),\ t \geq 0$	F	γ
1 i.e. $h_0(t)$	$\frac{1}{s}$	0
$h_a(t)$	$e^{-as}\frac{1}{s}$	0
$t^n, \quad n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	0
$e^{at}, \quad a \in \mathbb{C}$	$\frac{1}{s-a}$	$\mathrm{Re}\,(a)$
$e^{at}\sin(\omega t), \quad \omega \in \mathbb{R}$	$\frac{\omega}{(s-a)^2+\omega^2}$	0
$e^{at}\cos(\omega t), \quad \omega \in \mathbb{R}$	$\frac{s-a}{(s-a)^2+\omega^2}$	0

$h_a(t)$ for $a \geq 0$ is defined as follows: $h_a(t) := \begin{cases} 1 & t \geq a \geq 0, \\ 0 & t < a. \end{cases}$

If $f(t) \circ\!\!-\!\!\bullet\, F(s)$, then the following shifting theorems hold.

$I)$ $\quad h_a(t)f(t-a) \quad \circ\!\!-\!\!\bullet \quad e^{-sa}F(s)$ — Shifting in the original space / Multiplication with exponential function in the image space

$II)$ $\quad e^{at}f(t),\ a \in \mathbb{C} \quad \circ\!\!-\!\!\bullet \quad F(s-a)$ — Shifting in the image space/ Multiplication with exponential function in the original space

Exercise 1:

a) Which is the algebraic equation resulting from Laplace transformation of the initial value problem

$$u'' - 2u' + u = \sin(4t) + 2te^{-t}, \text{ for } t > 0, \qquad u(0) = 1, u'(0) = 0?$$

Please justify your answer by intermediate computations.
Compute the solution of the algebraic equation.

b) Let $F(s) = \dfrac{1}{s(s+1)^2}$ be the Laplace transform of the function

$$f : \mathbb{R}_0^+ \to \mathbb{R}, \quad f : t \mapsto f(t).$$

Determine $f(t)$.

Solution:

a) $u(t) \circ\!\!-\!\!\bullet\, U(s), \qquad u'(t) \circ\!\!-\!\!\bullet\, sU(s) - u(0) = sU(s) - 1,$
$u''(t) \circ\!\!-\!\!\bullet\, s^2U(s) - s - u'(0) = s^2U(s) - s,$
$\sin(4t) \circ\!\!-\!\!\bullet\, \dfrac{4}{s^2+16}, \qquad t \circ\!\!-\!\!\bullet\, \dfrac{1}{s^2}, \qquad e^{-t}t \circ\!\!-\!\!\bullet\, \dfrac{1}{(s+1)^2}.$

The IVP becomes

$$(s^2 - 2s + 1)U + 2 - s = \frac{4}{s^2+16} + \frac{2}{(s+1)^2}.$$

with the solution

$$U(s) = \frac{4}{(s^2+16)(s-1)^2} + \frac{2}{(s+1)^2(s-1)^2} + \frac{s-2}{(s-1)^2}$$

b) The partial fraction decomposition ansatz

$$\frac{as+b}{s^2+2s+1} + \frac{c}{s} = \frac{1}{s(s+1)^2}$$

returns

$cs^2 + 2cs + c + as^2 + bs = 1 \iff c = -a, \; -2c = b, \; c = 1.$

$$\frac{-s-2}{(s+1)^2} + \frac{1}{s} = \frac{-1}{(s+1)^2} - \frac{s+1}{(s+1)^2} + \frac{1}{s}$$

$$= \frac{-1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s} \bullet\!\!-\!\!\circ\, -te^{-t} - e^{-t} + 1.$$

Exercise 2:

a) For the following matrices $\boldsymbol{A}$ analyse the stability of the stationary point $(0,0)^T$ of the linear system $\boldsymbol{u}'(t) = A\,\boldsymbol{u}(t)$.

i) $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, **ii)** $A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, **iii)** $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

b) Consider the following linear system

$$\dot{\boldsymbol{u}}(t) = \begin{pmatrix} -3 & 0 & 3 \\ -1 & -\gamma & 1 \\ 3 & 0 & -3 \end{pmatrix} \boldsymbol{u}(t)\,.$$

Determine the stability behaviour of the stationary point $(0,0,0)^T$ depending on the parameter $\gamma \in \mathbb{R}$.

Sketch of solution:

a) **i)** $P(\lambda) = (1-\lambda)^2 + 1 = 0 \iff \lambda_{1,2} = 1 \pm i\,.$

There is at least one eigenvalue with positive real part. $\binom{0}{0}$ is an unstable stationary point.

ii) $P(\lambda) = (-1-\lambda)^2 = 0 \iff \lambda_{1,2} = -1\,.$

The real parts of all eigenvalues are negative. $\binom{0}{0}$ is an asymptotically stable stationary point.

iii) $P(\lambda) = \lambda^2 + 1 = 0 \iff \lambda_{1,2} = \pm i\,.$

There is no eigenvalue with positive real part. The eigenvalues with real part zero are simple. $\binom{0}{0}$ is a stable stationary point.

b) Characteristic polynomial: $P(\lambda) = (-\gamma - \lambda)[(3+\lambda)^2 - 9]\,.$

Eigenvalues: $\lambda_1 = -\gamma, \quad \lambda_2 = -3+3 = 0, \quad \lambda_3 = -3-3 = -6.$

$\gamma > 0 \iff \lambda_1,\, \lambda_3 < 0,\, \lambda_2 = 0$ simple eigenvalue: stable

$\gamma < 0 \iff \lambda_1 > 0$: unstable

$\gamma = 0 \implies \lambda_1 = 0,\, \lambda_2 = 0,\, \lambda_3 = -6$

Eigenspace corresponding to the double eigenvalue zero:

$$\begin{pmatrix} -3 & 0 & 3 \\ -1 & 0 & 1 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \iff v_3 = v_1$$

Eigenvectors:

$$v^{[1]} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{[2]} := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenspace of the eigenvalue zero with multiplicity two has dimension two. The zero solution is stable.

Dates of classes: 22.01.-26.01.2024