Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 6, Homework

Exercise 1:

Consider the following system of differential equations

$$\boldsymbol{u}'(t) = \begin{pmatrix} -3 & 2 \\ -8 & -3 \end{pmatrix} \boldsymbol{u}(t) + \begin{pmatrix} 20 \\ 20 \end{pmatrix} .$$

a) Determine a real fundamental system of the corresponding homogeneous system of differential equations .

b) With the help of an appropriate ansatz determine a particular solution of the inhomogeneous system and provide the general solution of the inhomogeneous differential equation.

Sketch of solution:

a)

$$\det \begin{pmatrix} -3-\lambda & 2 \\ -8 & -3-\lambda \end{pmatrix} = (-3-\lambda)^2 + 16.$$

The eigenvalues of the system matrix are found as follows

$$(-3-\lambda)^2 = -16 \Longleftrightarrow -3-\lambda = \pm 4i \Longrightarrow \lambda_{1,2} = -3 \pm 4i\,.$$

An eigenvector for $\lambda_1 = -3-4i$ is obtained as solution of the system of equations

$$\begin{pmatrix} 4i & 2 \\ -8 & 4i \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Any vector with $z_2 = -2iz_1$ satisfies the system. We may choose for example $(1, -2i)^T$. The complex conjugate vector is an eigenvector for $\lambda_2 = -3+4i$.

From this we get the complex fundamental system

$$\boldsymbol{z}^{[1]}(t) = e^{(-3+4i)t} \begin{pmatrix} 1 \\ 2i \end{pmatrix} , \quad \boldsymbol{z}^{[2]}(t) = e^{(-3-4i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix} .$$

A real fundamental system is given, for example, by $FM(t) = (\,\mathrm{Re}\,(\boldsymbol{z}^{[1]}(t))\,, \mathrm{Im}\,(\boldsymbol{z}^{[1]}(t))\,)$.

Due to $\boldsymbol{z}^{[1]}(t) = e^{-3t}(\cos(4t) + i\cdot\sin(4t)) \cdot \begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^{-3t} \begin{pmatrix} \cos(4t) + i\cdot\sin(4t) \\ 2i\cos(4t) - 2\sin(4t) \end{pmatrix}$,

one obtains

$$\boldsymbol{u}^{[1]}(t) = e^{-3t} \begin{pmatrix} \cos(4t) \\ -2\sin(4t) \end{pmatrix} , \quad \boldsymbol{u}^{[2]}(t) = e^{-3t} \begin{pmatrix} \sin(4t) \\ 2\cos(4t) \end{pmatrix} .$$

and thus the fundamental system $FM(t) := \left(\boldsymbol{u}^{[1]}(t),\, \boldsymbol{u}^{[2]}(t)\right)$.

Then the general solution of the homogeneous system reads $\boldsymbol{u}_h(t) = c_1\,\boldsymbol{u}^{[1]}(t) + c_2\,\boldsymbol{u}^{[2]}(t)$.

b) To solve the inhomogeneous system

$$\boldsymbol{u}'(t) = \begin{pmatrix} -3 & 2 \\ -8 & -3 \end{pmatrix} \boldsymbol{u}(t) + \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

we make the ansatz $\boldsymbol{u}^{[p]} = \begin{pmatrix} a \\ b \end{pmatrix}$ with constant numbers a, b and obtain

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -8 & -3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 20 \\ 20 \end{pmatrix}$$

$$\iff \begin{cases} -3a + 2b = -20 & \iff -9a + 6b = -60 \\ -8a - 3b = -20 & \implies -16a - 6b = -40, \end{cases}$$

Adding the last equations returns $-25a = -100$, thus $a = 4$.
Inserting a into any of the equations returns $b = -4$.
$\boldsymbol{u}^{[p]}(t) = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ is thus a particular solution of the inhomogeneous system. The general solution of the inhomogeneous system is

$$\boldsymbol{u}(t) = \boldsymbol{u}_h(t) + \boldsymbol{u}^{[p]}(t) = c_1 \boldsymbol{u}^{[1]}(t) + c_2 \boldsymbol{u}^{[2]}(t) + \boldsymbol{u}^{[p]}(t), \qquad c_1, c_2 \in \mathbb{R}.$$

Exercise 2)

Consider the system of differential equations

$$\boldsymbol{u}'(t) = \boldsymbol{A} \cdot \boldsymbol{u}(t) = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} \cdot \boldsymbol{u}(t).$$

a) Determine the general solution of the system.

b) Determine the solution $\boldsymbol{u}(t)$ of the corresponding initial value problem with

$$\boldsymbol{u}(0) = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

and compute for this solution $\lim\limits_{t\to\infty} \boldsymbol{u}(t)$.

c) Does the solution of the system from part a) converge to zero for $t \to \infty$ for every initial conditions? Justify your answer.

Solution:

a) Computation of the eigenvalues of $\boldsymbol{A}$:

$$P(\lambda) := \det \begin{pmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & 1 \\ 0 & 0 & -2-\lambda \end{pmatrix} = (-2-\lambda) \cdot \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (-2-\lambda) \cdot (\lambda^2 - 1).$$

$P(\lambda) = 0 \Longrightarrow \lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1.$

Computation of the eigenvectors:

$\lambda_1 = -2$:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 + 2v_3 \\ v_1 + 2v_2 + v_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

First row $-2\times$ second row: $-3v_2 = 0$.

Inserting $v_2 = 0$ into the first or second row: $v_3 = -v_1$.

For example $\boldsymbol{v}^{[1]} := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and from this $\boldsymbol{u}^{[1]}(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$\lambda_2 = -1$:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + 2v_3 \\ v_1 + v_2 + v_3 \\ -v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Third row $v_3 = 0$.

Inserting $v_3 = 0$ into the first or second row: $v_2 = -v_1$.

For example $\boldsymbol{v}^{[2]} := \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and thus $\boldsymbol{u}^{[2]}(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

$\lambda_3 = 1$:

$$\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 1 \\ 0 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 + v_2 + 2v_3 \\ v_1 - v_2 + v_3 \\ -3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Third row $v_3 = 0$.

Inserting $v_3 = 0$ into the first or second row: $v_2 = v_1$.

Thus for example we can choose $\boldsymbol{v}^{[3]} := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and

$$\boldsymbol{u}^{[3]}(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

.

The general solution is: $\boldsymbol{u}(t) = c_1\, \boldsymbol{u}^{[1]}(t) + c_2\, \boldsymbol{u}^{[2]}(t) + c_3\, \boldsymbol{u}^{[3]}(t)$.

b)

$$\boldsymbol{u}(0) = c_1\, e^0 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\, e^0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3\, e^0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$\iff \begin{cases} c_1 + c_2 + c_3 = 3 \\ \quad -c_2 + c_3 = -1 \\ \qquad\quad -c_1 = -2 \Rightarrow c_1 = 2 \end{cases}$$

New system: $\begin{cases} c_2 + c_3 = 1 \\ -c_2 + c_3 = -1 \\ c_1 = 2 \end{cases}$.

Summing up the first two rows of the system returns: $c_3 = 0$ and from this it follows $c_2 = 1$. The solution of the initial value problem is:

$$\boldsymbol{u}(t) = 2 \cdot \boldsymbol{u}^{[1]}(t) + \boldsymbol{u}^{[2]}(t) = 2e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} .$$

$\lim_{t \to \infty} \boldsymbol{u}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

c) No, the solution from a)

$$\begin{aligned} \boldsymbol{u}(t) &= c_1 \cdot \boldsymbol{u}^{[1]}(t) + c_2 \cdot \boldsymbol{u}^{[2]}(t) + c_3 \boldsymbol{u}^{[3]}(t) \\ &= c_1\, e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2\, e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3\, e^t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

does not converge to zero for arbitrary initial values. It converges to the zero solution if and only if the initial values are such that c_3 vanishes!

Exercise 3:

Consider the linear differential equation of second order

$$u''(t) + \frac{7}{t}u'(t) + \frac{9}{t^2}u(t) \;=\; 0.$$

a) With the help of the ansatz: $u_0(t) = t^k$, determine a solution of the differential equation.

b) With the help of te reduction ansatz $\hat{u}(t) = u_0(t)\cdot w(t)$ find another solution of the differential equation and provide the general solution of the differential equation.

c) Compute the solution of the boundary problem

$$u''(t) + \frac{7}{t}u'(t) + \frac{9}{t^2}u(t) \;=\; 0, \qquad 1 < t < e^{\frac{1}{3}}, \qquad u(1) = 0,\, u(e^{\frac{1}{3}}) = 1.$$

d) Can you also calculate a solution of the following boundary value problem?

$$u''(t) + \frac{7}{t}u'(t) + \frac{9}{t^2}u(t) \;=\; 0, \qquad 1 < t < e^{\frac{1}{3}}, \qquad u(1) = 0,\, u'(e^{\frac{1}{3}}) = 1.$$

Solution:

a) The ansatz $u_0(t) = t^k$ returns for $t \neq 0$

$$k(k-1) + 7k + 9 \;=\; k^2 + 6k + 9 \;=\; (k+3)^2 \overset{!}{=} 0\,.$$

We obtain just one solution $u_0(t) = t^{-3}$ (up to multiplication by a constant). Since the space of solutions has dimension two, we do not find a fundamental system.

b) Inserting the reduction ansatz $\hat{u}(t) = u_0(t)\cdot w(t)$ into the differential equation returns as in the lecture

$$(u_0w)'' + \frac{7}{t}(u_0w)' + \frac{9}{t^2}(u_0w) = 0$$

$$\Rightarrow \qquad (u_0''w + 2u_0'w' + u_0w'') + \frac{7}{t}(u_0'w + u_0w') + \frac{9}{t^2}(u_0w) = 0$$

$$\Rightarrow \qquad u_0w'' + (2u_0' + \frac{7}{t}u_0)w' + \underbrace{(u_0'' + \frac{7}{t}u_0' + \frac{9}{t^2}u_0)}_{=0}w = 0$$

$$\Rightarrow \qquad t^{-3}w'' + (-6t^{-4} + \frac{7}{t}t^{-3})w' = 0 \qquad \overset{t\neq 0}{\Longleftrightarrow} w'' + \frac{1}{t}w' = 0$$

$$\overset{y=w'}{\Longrightarrow} \qquad y' = -\frac{1}{t}y.$$

This is a separable differential equation in $y(t)$

$$\frac{dy}{y} = -\frac{dt}{t} \Longrightarrow \ln(|y|) = \; -\ln(|t|) + k \Longrightarrow y(t) = \frac{c}{t} = w'(t)$$

$$w(t) = \; c\ln(t) + \tilde{c} \qquad \text{and thus for example } w(t) = \ln(t).$$

With this w we get from our ansatz $\hat{u}(t) = \; w(t)u_0(t) = \frac{\ln(t)}{t^3}$.

The general solution is

$$u(t) = \; c_1\frac{1}{t^3} \; + c_2\frac{\ln(t)}{t^3}, \qquad c_1,\, c_2 \in \mathbb{R}.$$

c) The boundary values require:

$u(1) = c_1 \frac{1}{1^3} + c_2 \frac{\ln(1)}{1^3} = c_1 = 0 \Longrightarrow u(t) = c_2 \frac{\ln(t)}{t^3}$

and

$u(e^{\frac{1}{3}}) = c_2 \frac{\ln\left(e^{\frac{1}{3}}\right)}{\left(e^{\frac{1}{3}}\right)^3} = c_2 \frac{\frac{1}{3}\ln(e)}{e^1} = 1 \Longrightarrow c_2 = 3e^1$

From this we obtain the unique solution $u(t) = 3e \frac{\ln(t)}{t^3}$.

d) From $u(1) = 0$ it follows again $c_1 = 0$ and hence $u(t) = c_2 \frac{\ln(t)}{t^3}$, thus

$u'(t) = c_2 \left(\frac{1}{t^4} - 3 \frac{\ln(t)}{t^4} \right) = \frac{c_2}{t^4} \left(1 - 3\ln(t)\right)$.

For every value of c_2, one thus obtains $u'(e^{\frac{1}{3}}) = \frac{c_2}{t^4} \left(1 - 3 \cdot \frac{1}{3}\right) = 0$.

The boundary condition $u'(e^{\frac{1}{3}}) = 1$ can thus not be satisfied. The boundary value problem has no solution.

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