Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 6, Exercise class

Exercise 1: For each of the following matrices

$$\boldsymbol{A}^{[1]} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \qquad \boldsymbol{A}^{[2]} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}, \qquad \boldsymbol{A}^{[3]} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \boldsymbol{A}^{[4]} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix},$$

determine a real fundamental system of the solution space of

$$\boldsymbol{y}'(t) = \boldsymbol{A}^{[k]}\,\boldsymbol{y}(t)\,, \qquad k = 1,2,3,4.$$

Solution:

$\boldsymbol{A}^{[1]} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

The eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ of the system matrix can be read from the diagonal.

We get an eigenvector for $\lambda_1 = 1$ as solution to the system of equations

$$\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We get an eigenvector for $\lambda_2 = 3$ as solution to the system

$$\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftarrow \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From this we obtain the fundamental system $\left\{ e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\boldsymbol{A}^{[2]} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$

The eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 3$ of the system matrix can be read from the diagonal.

The eigenvectors for $\lambda_{1,2} = 3$ have to satisfy the following system of equations:

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow v_2 = 0$$

Thus there is only one eigenvector direction $\boldsymbol{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We have to compute a generalized eigenvector

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longleftarrow \boldsymbol{w} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$

From this we obtain the fundamental system

$$\left\{ e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t}(t\,\boldsymbol{v} + \boldsymbol{w}) \right\} = \left\{ e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t} \begin{pmatrix} t \\ 0.5 \end{pmatrix} \right\}.$$

$\boldsymbol{A}^{[3]} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

The eigenvalues of the system matrix $\lambda_1 = 3 = \lambda_2 = 3$ can be once again read from the diagonal.

The eigenvectors for $\lambda_{1,2} = 3$ have to satisfy the following system of equations:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore we may choose any two linearly independent eigenvector directions and obtain, for example with the canonical eigenvectors, the fundamental system

$$\left\{ e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

$\boldsymbol{A}^{[4]} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

We obtain the eigenvalues of the system matrix from the condition $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 + 9 = 0$.

Hence $\lambda_1 = -3i$ and $\lambda_2 = 3i$.

The eigenvectors for $\lambda_1 = -3i$ are obtained as solution of the of the system

$$\begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Any vector with $z_1 = iz_2$ fulfills the system. Thus for example we can choose $(i, 1)^T$. The complex conjugate vector is an eigenvector for $\lambda_2 = 3i$.

Thuse we get the complex fundamental system

$$\boldsymbol{z}^{[1]}(t) = e^{3it} \begin{pmatrix} -i \\ 1 \end{pmatrix}, \quad \boldsymbol{z}^{[2]}(t) = e^{-3it} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

A real fundamental system is given for example by $\{ \operatorname{Re}(\boldsymbol{z}^{[1]}(t)), \operatorname{Im}(\boldsymbol{z}^{[1]}(t)) \}$.

$$\text{With} \quad \boldsymbol{z}^{[1]}(t) = e^{3it} \begin{pmatrix} -i \\ 1 \end{pmatrix} = (\cos(3t) + i\sin(3t)) \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -i\cos(3t) + \sin(3t) \\ \cos(3t) + i\sin(3t) \end{pmatrix}$$

the fundamental system is

$$\left\{ \begin{pmatrix} \sin(3t) \\ \cos(3t) \end{pmatrix}, \begin{pmatrix} -\cos(3t) \\ \sin(3t) \end{pmatrix} \right\}.$$

Exercise 2) Consider the system of differential equations

$$\boldsymbol{u}' = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \boldsymbol{u} + \begin{pmatrix} 4t \\ t \end{pmatrix} \qquad t \geq 0.5.$$

a) Show that

$$\boldsymbol{U}(t) := \begin{pmatrix} t^{-2} & t \\ -2t^{-2} & t \end{pmatrix}$$

is a fundamental system of the corresponding homogeneous system of differential equations .

b) Determine the general solution of the inhomogeneous problem.

c) Determine the solution of the corresponding initial value problem with initial values $\boldsymbol{u}(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Solution:

a) It holds

$$\det(\boldsymbol{U}(t)) = t^{-1} + 2t^{-1} \neq 0 \qquad \forall t \geq 0.5.$$

Thus the functions

$$\boldsymbol{u}_1(t) = \begin{pmatrix} t^{-2} \\ -2t^{-2} \end{pmatrix}, \quad \boldsymbol{u}_2(t) = \begin{pmatrix} t \\ t \end{pmatrix}$$

determine a fundamental system if they are solutions of the homogeneous system. It holds

$$\boldsymbol{u}_1' = \begin{pmatrix} \frac{-2}{t^3} & \frac{4}{t^3} \end{pmatrix}^T$$

$$\frac{1}{t}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \boldsymbol{u}_1 = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{t^2} \\ \frac{-2}{t^2} \end{pmatrix} = \frac{1}{t}\begin{pmatrix} \frac{-2}{t^2} \\ \frac{4}{t^2} \end{pmatrix}$$

$\boldsymbol{u}_1$ is thus a solution of the homogeneous problem. For $\boldsymbol{u}_2$ one computes analogously

$$\boldsymbol{u}_2' = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$$

$$\frac{1}{t}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \boldsymbol{u}_2 = \frac{1}{t}\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix} = \frac{1}{t}\begin{pmatrix} t \\ t \end{pmatrix}$$

$\boldsymbol{u}_2$ is thus another solution of the homogeneous differential equation , hence $\boldsymbol{U}(t)$ is a fundamental matrix of the homogeneous system.

b) In order to determine the general solution of the inhomogeneous problem, we make the ansatz $\boldsymbol{u}_p(t) := \boldsymbol{U}(t) \cdot \mathbf{c}(t)$ (variation of constants). This inserted into the differential equation returns the condition

$$\boldsymbol{U}(t) \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 4t \\ t \end{pmatrix}$$

$$\iff \begin{pmatrix} t^{-2} & t \\ -2t^{-2} & t \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} \frac{c_1'(t)}{t^2} + tc_2'(t) \\ \frac{-2c_1'(t)}{t^2} + tc_2'(t) \end{pmatrix} = \begin{pmatrix} 4t \\ t \end{pmatrix}$$

$$\iff \begin{pmatrix} \frac{3c_1'(t)}{t^2} \\ \frac{-2c_1'(t)}{t^2} + tc_2'(t) \end{pmatrix} = \begin{pmatrix} 3t \\ t \end{pmatrix} \iff \begin{pmatrix} c_1'(t) \\ -2t + tc_2'(t) \end{pmatrix} = \begin{pmatrix} t^3 \\ t \end{pmatrix}$$

$$\iff \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} t^3 \\ 3 \end{pmatrix} \Longleftarrow \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} \frac{t^4}{4} \\ 3t \end{pmatrix}$$

From this we get the particular solution

$$\boldsymbol{u}_p(t) := \boldsymbol{U}(t) \cdot \mathbf{c}(t) = \begin{pmatrix} t^{-2} & t \\ -2t^{-2} & t \end{pmatrix} \begin{pmatrix} \frac{t^4}{4} \\ 3t \end{pmatrix} = t^2 \begin{pmatrix} \frac{13}{4} \\ \frac{5}{2} \end{pmatrix}$$

Thus the general solution of the inhomogeneous problem reads as follows

$$\boldsymbol{u}(t) = k_1 \begin{pmatrix} t^{-2} \\ -2t^{-2} \end{pmatrix} + k_2 \begin{pmatrix} t \\ t \end{pmatrix} + t^2 \begin{pmatrix} \frac{13}{4} \\ \frac{5}{2} \end{pmatrix}$$

c) Inserting the initial values returns:

$$\begin{pmatrix} k_1 + k_2 + \frac{13}{4} \\ -2k_1 + k_2 + \frac{5}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow k_1 = \frac{1}{12},\ k_2 = -\frac{7}{3}.$$

Dates of classes: 08.01-12.01.2024