Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I
for Students of Engineering Sciences
Sheet 5, Homework

Exercise 1)

Consider the following fourth-order differential equation

$$u^{(4)}(t) + a_3 u^{'''}(t) + a_2 u^{''}(t) + a_1 u'(t) + a_0 u(t) = 0 \tag{1}$$

with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For each of the following sets of functions, determine whether they can be (with suitable coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$) a fundamental system for the solution space of the differential equation.

Justify your answers.

a) $M_1 := \{u_1(t) = e^t,\, u_2(t) = e^{5t},\, u_3(t) = e^{9t}\}$.

b) $M_2 := \{u_1(t) = e^t,\, u_2(t) = e^{it},\, u_3(t) = e^{2t},\, u_4(t) = e^{2it}\}$.

c) $M_3 := \{u_1(t) = 1,\, u_2(t) = t,\, u_3(t) = e^{2t},\, u_4(t) = e^{-2t}\}$.

d) $M_4 := \{u_1(t) = e^t,\, u_2(t) = \sin(2t),\, u_3(t) = e^{-2it},\, u_4(t) = e^{2it}\}$.

Solution:

a) Since the space of solutions has dimension four, M_1 cannot be a fundamental system for (1).

b) Complex solutions of linear differential equations with constant real coefficients always occur in conjugated complex pairs! Therefore M_2 cannot be a fundamental system for (1).

c) M_3 is a fundamental system for (1) with appropriate coefficients.

Not required from the students: the characteristic polynomial would be
$P(\lambda) = \lambda^2(\lambda^2 - 4)$, the differential equation $y^{''''}(t) - 4y^{''}(t) = 0$.

d) M_4 cannot be a fundamental system, since it holds:
$e^{2it} - e^{-2it} = 2i\sin(2t)$. Thus the space spanned by the functions in M_4 has only dimension three.

Exercise 2)

Consider the differential equation

$$u''(t) + 9u(t) = b(t)$$

a) Determine a real representation for the general solution of the corresponding homogeneous differential equation.

b) Compute the solutions of the differential equation for the inhomogeneities

i) $b(t) = 5e^{-t}$, **ii)** $b(t) = 5\sin(2t)$, **iii)** $b(t) = 5\sin(3t)$.

c) Determine the solution of the corresponding initial value problems for the initial values

$$u(0) = u'(0) = 0\,.$$

In each case check whether the solutions are bounded for $t \geq 0$ and whenever possible provide upper bounds for $|u(t)|,\, t \geq 0$.

Solution:

a) Characteristic polynomial: $P(\lambda) = \lambda^2 + 9 \stackrel{!}{=} 0 \Longrightarrow \lambda_{1,2} = \pm 3i$.

Basis of the solution space: $\{z^{[1]}(t) = e^{3it}, z^{[2]}(t) = e^{-3it}\}$.

With $u_1(t) := \mathrm{Re}\,(z^{[1]}(t)) = \dfrac{z^{[1]}(t) + z^{[2]}(t)}{2} = \cos(3t)$

and $u_2(t) := \mathrm{Im}\,(z^{[1]}(t)) = \dfrac{z^{[1]}(t) - z^{[2]}(t)}{2i} = \sin(3t)$

one obtains a real basis of the solution space and the general solution

$$u_h(t) = c_1\cos(3t) + c_2\sin(3t)\,.$$

b) **i)** $b(t) = 5e^{-t}$. Ansatz: $u_p(t) = k \cdot e^{-t}$.

Inserting into the differential equation returns

$$k \cdot e^{-t} + 9k \cdot e^{-t} \stackrel{!}{=} 5 \cdot e^{-t} \Longrightarrow k = \frac{1}{2}.$$

$$u(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{1}{2}e^{-t}.$$

ii) $b(t) = 5\sin(2t)$. Ansatz: $u_p(t) = a \cdot \cos(2t) + b \cdot \sin(2t)$.

Alternatively $b(t) = \frac{5}{2i}(e^{2it} + e^{-2it})$. Ansatz: $u_p(t) = c_1 e^{2it} + c_2 e^{-2it}$.

Inserting the real ansatz into the differential equation returns

$$-4a \cdot \cos(2t) - 4b \cdot \sin(2t) + 9a \cdot \cos(2t) + 9b \cdot \sin(2t) \stackrel{!}{=} 5\sin(2t) \Longrightarrow a = 0,\, b = 1\,.$$

$$u(t) = c_1\cos(3t) + c_2\sin(3t) + \sin(2t).$$

iii) $b(t) = 5\sin(3t) = \frac{5}{2i}(e^{3it} + e^{-3it})$

Since $\sin(3t)$ solves the homogeneous differential equation , we make the ansatz:

$u_p(t) = t(a\cdot\cos(3t) + b\cdot\sin(3t))$.

Alternative ansatz: $u_p(t) = t(c_1 e^{3it} + c_2 e^{-3it})$.

Then it holds:

$u_p'(t) = (a\cdot\cos(3t) + b\cdot\sin(3t)) + t(-3a\cdot\sin(3t) + 3b\cdot\cos(3t))$.

$u_p''(t) = (-6a\cdot\sin(3t) + 6b\cdot\cos(3t)) + t(-9a\cdot\cos(3t) - 9b\cdot\sin(3t))$.

Inserting into the differential equation yields

$$\begin{aligned} &u_p'' + 9u_p \\ &= -6a\cdot\sin(3t) + 6b\cdot\cos(3t) + t(-9a\cdot\cos(3t) - 9b\cdot\sin(3t)) + 9t(a\cdot\cos(3t) + b\cdot\sin(3t)) \\ &= -6a\cdot\sin(3t) + 6b\cdot\cos(3t) \stackrel{!}{=} 5\sin(3t) \Rightarrow b = 0,\ -6a = 5, \Rightarrow u_p(t) = -\frac{5}{6}t\cos(3t). \end{aligned}$$

$$u(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{5}{6}t\cos(3t).$$

c) **i)** $b(t) = 5e^{-t}$, $\qquad u(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{1}{2}e^{-t}$.

$u(0) = c_1 + \frac{1}{2} \stackrel{!}{=} 0 \Longrightarrow c_1 = -\frac{1}{2}$.

$u'(0) = 3c_2\cos(0) - \frac{1}{2} \stackrel{!}{=} 0 \Longrightarrow c_2 = \frac{1}{6}$

$$u(t) = c_1\cos(3t) + c_2\sin(3t) + \frac{1}{2}e^{-t} = -\frac{1}{2}\cos(3t) + \frac{1}{6}\sin(3t) + \frac{1}{2}e^{-t}.$$

$$|u(t)| \le |c_1| + |c_2| + \frac{1}{2} = \frac{7}{6}, \qquad \forall t \ge 0\,.$$

ii) $b(t) = 5\sin(2t)$, $\qquad u(t) = c_1\cos(3t) + c_2\sin(3t) + \sin(2t)$.

$u(0) = c_1 \stackrel{!}{=} 0 \Longrightarrow c_1 = 0$.

$u'(0) = 3c_2\cos(0) + 2\cos(0) = 3c_2 + 2 \stackrel{!}{=} 0 \Longrightarrow c_2 = -\frac{2}{3}$

$$u(t) = -\frac{2}{3}\sin(3t) + \sin(2t).$$

$$|u(t)| \le |c_1| + |c_2| + 1 = \frac{5}{3}, \qquad \forall t \ge 0\,.$$

iii) $b(t) = 5\sin(3t)$, $\qquad u(t) = c_1\cos(3t) + c_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

$u(0) = c_1 \stackrel{!}{=} 0 \Longrightarrow c_1 = 0$.

$u'(0) = 3c_2\cos(0) - \frac{5}{6}\cos(0) \stackrel{!}{=} 0 \Longrightarrow c_2 = \frac{5}{18}$

$$u(t) = \frac{5}{18}\sin(3t) - \frac{5}{6}t\cos(3t).$$

Here one gets for example for $t_k = 2k\pi, k \in \mathbb{N}$

$$u(2k\pi) = c_2\sin(6k\pi) - \frac{5}{6}(2k\pi)\cos(6k\pi) = -\frac{5k\pi}{3}.$$

And thus $\lim\limits_{k\to\infty} |u(2k\pi)| = \infty$.

The solution is unbounded (resonance case).

Exercise 3) Somewhat more demanding.

We look for a particular solution u_p of the inhomogeneous differential equation

$$\mathcal{L}_0[u] := \sum_{k=0}^{m} a_k u^{(k)}(t) = b(t) = b_0 e^{\alpha t}, \qquad a_m = 1, a_k \in \mathbb{R},\, 0 \neq b_0 \in \mathbb{R},\, \alpha \in \mathbb{C}.$$

a) Prove that the ansatz $u_p(t) = Be^{\alpha t}$, $B \in \mathbb{C}$ is successful if and only if α is not a root of the characteristic polynomial

$$P_0(\lambda) := \sum_{k=0}^{m} a_k \lambda^k.$$

b) Prove that the ansatz $u_p(t) = Bte^{\alpha t}$, $B \in \mathbb{C}$ is successful if α is a simple root of the characteristic polynomial

$$P_0(\lambda) := \sum_{k=0}^{m} a_k \lambda^k.$$

Hint: Use the factorization from page 40 of the lecture.

c) Let now α be a root of the characteristic polynomial with multiplicity $l \in \mathbb{N},\ l \geq 2$. Thus

$$P_0(\lambda) = P_l(\lambda)(\lambda - \alpha)^l,\ P_l(\alpha) \neq 0$$

and

$$\mathcal{L}_0[u] := \mathcal{L}_l\left[\left(\frac{d}{dt} - \alpha\right)^l u\right].$$

Prove that $u_p(t) := Bt^l e^{\alpha t}$, $B \in \mathbb{C}$ is an appropriate ansatz for a fundamental solution of the differential equation.

Hints:

Define P_j and $\mathcal{L}_j$ by

$P_0(\lambda) = P_j(\lambda)(\lambda - \alpha)^j$, $\mathcal{L}_0[u] = \mathcal{L}_j\left[\left(\frac{d}{dt} - \alpha\right)^j u\right]$, $\qquad j = 0, 1, 2, \ldots, l.$

α is a root of multiplicity $(l-j)-$ of P_j. In particular it is not a zero of P_l.

Show

$$\mathcal{L}_0[Bt^l e^{\alpha t}] = \frac{l!}{(l-j)!} B\mathcal{L}_j[t^{l-j} e^{\alpha t}]$$

by induction and using the factorization method on page 40 of the lecture.

Solution:

a) To be fulfilled is $\mathcal{L}_0[Be^{\alpha t}] = b_0 e^{\alpha t}$ with given $b_0 \neq 0$.

$\mathcal{L}_0[Be^{\alpha t}] = BP_0(\alpha)e^{\alpha t} \overset{!}{=} b_0 e^{\alpha t}$

This equation for B can be solved by $B = \frac{b_0}{P_0(\alpha)}$ iff $P_0(\alpha) \neq 0$.

b) Let α be a simple root of P_0, then it holds

$$P_0(\lambda) = P_1(\lambda)(\lambda - \alpha), \qquad P_1(\alpha) \neq 0$$

and

$$\mathcal{L}_0[u] := \mathcal{L}_1\left[\left(\frac{d}{dt} - \alpha\right) u\right], \qquad \mathcal{L}_1[e^{\alpha t}] \neq 0.$$

To be fulfilled is

$$\mathcal{L}_0[Bte^{\alpha t}] = \mathcal{L}_1\left[\left(\frac{d}{dt} - \alpha\right)^1 Bte^{\alpha t}\right] = \mathcal{L}_1[Be^{\alpha t} + \alpha Bte^{\alpha t} - \alpha Bte^{\alpha t}] = B\mathcal{L}_1[e^{\alpha t}] = B\,P_1(\alpha)e^{\alpha t} \overset{!}{=} b_0 e^{\alpha t}.$$

This is satisfied for $B = \frac{b_0}{P_1(\alpha)}$.

c) Let α be a zero of P_0 of multiplicity l. P_j and $\mathcal{L}_j$ are defined as above.
Claim
$$\mathcal{L}_0[Bt^l e^{\alpha t}] = \frac{l!}{(l-j)!} B\mathcal{L}_j[t^{l-j}e^{\alpha t}], \qquad j = 1, 2, \dots, l.$$

Proof by Induction:
$j = 1$

$$\begin{aligned}\mathcal{L}_0[Bt^l e^{\alpha t}] &= \mathcal{L}_1\left[\left(\frac{d}{dt} - \alpha\right) Bt^l e^{\alpha t}\right] \\ &= \mathcal{L}_1[B(lt^{l-1} + t^l\alpha - t^l\alpha)e^{\alpha t}] = \frac{l!}{(l-1)!}B\mathcal{L}_1[t^{l-1}e^{\alpha t}].\end{aligned}$$

Induction hypothesis: For an arbitrary $j \in \mathbb{N}, 1 \leq j \leq l-1$ it holds:

$$\mathcal{L}_0[Bt^l e^{\alpha t}] = \frac{l!}{(l-j)!}B\mathcal{L}_j[t^{l-j}e^{\alpha t}]$$

Then it also holds

$$\begin{aligned}\mathcal{L}_0[Bt^l e^{\alpha t}] &= \frac{l!}{(l-j)!}B\mathcal{L}_j[t^{l-j}e^{\alpha t}] \\ &= \frac{l!}{(l-j)!}B\mathcal{L}_{j+1}\left[\left(\frac{d}{dt} - \alpha\right) t^{l-j}e^{\alpha t}\right] = \frac{l!}{(l-j)!}B\mathcal{L}_{j+1}[(l-j)t^{l-(j+1)} + \alpha t^{l-j} - \alpha t^{l-j})e^{\alpha t}] \\ &= \frac{l!}{(l-j)!}B\mathcal{L}_{j+1}[(l-j)t^{l-(j+1)}e^{\alpha t}] = \frac{l!}{(l-(j+1))!}B\mathcal{L}_{j+1}[t^{l-(j+1)}e^{\alpha t}]\end{aligned}$$

The claim then holds for $j = 1, 2, \dots l$. In particular even for $j = l$:

$$\mathcal{L}_0[Bt^l e^{\alpha t}] = \frac{l!}{(l-l)!}B\mathcal{L}_l[t^{l-l}e^{\alpha t}] = l!B\mathcal{L}_l[e^{\alpha t}].$$

Since $\mathcal{L}_l[e^{\alpha t}] \neq 0$, we can choose $B = \frac{b_0 e^{\alpha t}}{l!\mathcal{L}_l[e^{\alpha t}]}$ and with $u_p(t) = Bt^l e^{\alpha t}$ obtain a particular solution of the differential equation.

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