Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I
for Students of Engineering Sciences
Sheet 5, Exercise class

Exercise 1: Determine the general solutions of the following linear differential equations

a) $u^{(3)} - 3\,u' - 2\,u = e^{-2t}$.

b) $u^{(3)} - 3\,u' - 2\,u = e^{2t}$.

c) $u^{(3)} - 3\,u' - 2\,u = te^{-2t}$.

d) $u^{(3)} - 3\,u' - 2\,u = 7e^{2t} - 5\,e^{-2t}$.

Hint: For the particular solution of the inhomogeneous problem you may use a special ansatz.

Solution:

We first solve the corresponding homogeneous differential equation

$$u^{(3)} - 3\,u' - 2\,u = 0$$

with the characteristic polynomial

$P(\lambda) = \lambda^3 - 3\lambda - 2 = (\lambda+1)(\lambda^2 - \lambda - 2) = (\lambda+1)^2(\lambda-2) = 0\,.$

$P(\lambda) = 0 \iff \lambda_{1,2} = -1\,,\ \lambda_3 = 2.$

From this we obtain as basis of the space of solutions: $u_1(t) = e^{-t}$, $u_2(t) = t\,e^{-t}$, $u_3(t) = e^{2t}$.

We still need to compute a particular/special solution of the inhomogeneous differential equations respectively .

a) $u^{(3)} - 3\,u' - 2\,u = e^{-2t}$

Inhomogeneity: (polynomial of degree zero) $\cdot e^{\mu t}$ where μ is not a root of the characteristic polynomial. Thus with a constant $B \in \mathbb{R}$ we make the ansatz

$$u_{p,1}(t) := B\,e^{-2t}.$$

Inserting into the differential equation returns the requirement

$$u^{(3)} - 3\,u' - 2\,u = e^{-2t}\,[-8(B) - 3(-2B) - 2B] = e^{-2t}$$

$$\Longrightarrow -4B = 1 \Longrightarrow u_{p,1} = -\frac{1}{4}\,e^{-2t}\,.$$

General solution:

$$u(t) = u_{p,1}(t) + u_h(t) = -\frac{1}{4}\,e^{-2t} + (c_1 + c_2\,t)\,e^{-t} + c_3\,e^{2t}\,.$$

b) $u^{(3)} - 3\,u' - 2\,u = e^{2t}$

Inhomogeneity: (polynomial of degree zero) $\cdot e^{\mu t}$ where μ is a simple root of the characteristic polynomial. Here with a constant $a \in \mathbb{R}$ we make the ansatz

$$u_{p,2}(t) := a t\,e^{2t}.$$

Inserting into the differential equation returns the constraint

$$u^{(3)} - 3\,u' - 2\,u = e^{2t}\,[(8at + 8a + 4a) - 3(2at + a) - 2(at)\,] = e^{2t}$$
$$\Longrightarrow\; 0\cdot t + 9a = 1 \;\Longrightarrow\; u_{p,2} = \frac{t}{9}\,e^{2t}\,.$$

General solution:

$$u(t) = u_{p,2}(t) + u_h(t) = \frac{t}{9}\,e^{2t} + (c_1 + c_2\,t)\,e^{-t} + c_3\,e^{2t}\,.$$

c) $u^{(3)} - 3\,u' - 2\,u = te^{-2t}\,.$

Inhomogeneity: (polynomial of degree one) $\cdot e^{\mu t}$ where μ is not a root of the characteristic polynomial. According to page 43 of the lecture, the ansatz is
$u_{p,3}(t) := (B_1t^1 + B_0t^0)\,e^{-2t}$. In order to simplify the notation we choose $u_{p,3}(t) := (at + b)\,e^{-2t}$.

$$\begin{aligned}
u_{p,3}(t) &:= (at + b)\,e^{-2t}\\
u'_{p,3}(t) &= (a - 2at - 2b)\,e^{-2t},\\
u''_{p,3}(t) &= (-2a - 2a + 4at + 4b)\,e^{-2t} = (-4a + 4at + 4b)\,e^{-2t},\\
u^{(3)}_{p,3}(t) &= (4a + 8a - 8at - 8b)\,e^{-2t} = (12a - 8at - 8b)\,e^{-2t}.
\end{aligned}$$

Inserting into the differential equation returns the constraint

$$u^{(3)} - 3\,u' - 2\,u = e^{-2t}\,[\,12a - 8at - 8b - 3(a - 2at - 2b) - 2(at + b)] = te^{-2t}$$
$$\Longrightarrow \begin{cases} t^0: & 12a - 8b - 3a + 6b - 2b = 0 \;\Longrightarrow 9a - 4b = 0 \Longrightarrow b = \frac{9}{4}a\\ t: & -8a + 6a - 2a = 1 \;\Longrightarrow a = -\frac{1}{4}\end{cases}$$
$$u_{p,3} = (-\frac{t}{4} - \frac{9}{16})e^{-2t}\,.$$

General solution:

$$u(t) = u_{p,3}(t) + u_h(t) = -\frac{t}{4} - \frac{9}{16}\,e^{-2t} + (c_1 + c_2\,t)\,e^{-t} + c_3\,e^{2t}\,.$$

d) Due to the linearity of the problem, one gets a particular solution of the new problem as linear combination of the two particular $u_{p,1}(t)$ and $u_{p,2}(t)$. Thus the general solution is

$$u(t) = 7u_{p,2}(t) - 5u_{p,1}(t) + u_h(t) = \frac{7}{9}\,te^{2t} + \frac{5}{4}\,e^{-2t} + (c_1 + c_2\,t)\,e^{-t} + c_3\,e^{2t}\,.$$

Exercise 2)

a) Determine a real representation of the general solution of the differential equation

$$u^{(3)}(t) + u''(t) + 3u'(t) - 5u(t) = 0 .$$

b) Determine the general solutions of the differential equations :

i) $u^{(3)}(t) + u''(t) + 3u'(t) - 5u(t) = 10,$ **ii)** $u^{(3)}(t) + u''(t) + 3u'(t) - 5u(t) = e^t.$

Solution:

a) Characteristic polynomial: $P(\lambda) = \lambda^3 + \lambda^2 + 3\lambda - 5$.
$\lambda_1 = 1$ is a zero of P . Polynomial division yields
$P(\lambda) = (\lambda - 1)(\lambda^2 + 2\lambda + 5) = (\lambda - 1)\left((\lambda+1)^2 + 4\right)$.
The zeroes of P are: $\lambda_1 = 1,\ \lambda_2 = -1 + 2i,\ \lambda_3 = -1 - 2i$.
Complex fundamental system:
$u_1(t) = e^t,$
$z_2(t) = e^{(-1+2i)t} = e^{-t}e^{2it} = e^{-t}\cos(2t) + ie^{-t}\sin(2t)$
$z_3(t) = e^{(-1-2i)t} = e^{-t}e^{-2it}\cdot = e^{-t}\cos(-2t) + ie^{-t}\sin(-2t) = e^{-t}\cos(2t) - ie^{-t}\sin(2t)$.

With

$$u_2(t) := \frac{z_2(t) + z_3(t)}{2} = \mathrm{Re}\,(e^{(-1+2i)t}) \text{ and } u_3(t) := \frac{z_2(t) - z_3(t)}{2i} = \mathrm{Im}\,(e^{(-1+2i)t})$$

we obtain a real basis of the solution space:
$u_1(t) = e^t,\ u_2(t) = e^{-t}\cdot\cos(2t),\ u_3(t) = e^{-t}\cdot\sin(2t)$.
General solution: $u(t) = c_1e^t + c_2\,e^{-t}\cdot\cos(2t) + c_3\,e^{-t}\cdot\sin(2t)$.

b) The general solution of the homogeneous differential equation is known from Part a). For particular solutions of each of the inhomogeneous equations we make a special ansatz.

i) Inhomogeneity: polynomial of degree zero times $e^{0\cdot t}$, where 0 is not a zero of the characteristic polynomial.
Ansatz: $u_p =$ polynomial of degree zero times $e^{0\cdot t} = B$.
Inserting into the differential equation yields:
$-5B = 10 \iff u_p = B = -2 \qquad u(t) = c_1e^t + c_2\,e^{-t}\cdot\cos(2t) + c_3\,e^{-t}\cdot\sin(2t) - 2$.

ii) Inhomogeneity: polynomial of degree zero times $e^{1\cdot t}$, where 1 is a zero of the characteristic polynomial.
Ansatz: $u_p = t\cdot$ polynomial of degree zero times $e^{1\cdot t} = B\cdot t^1\cdot e^t$, such that

$$u_p'(t) = B\cdot(1+t)\cdot e^t, \qquad u_p''(t) = B\cdot(2+t)\cdot e^t, \qquad u_p^{(3)}(t) = B\cdot(3+t)\cdot e^t,$$

Inserting into the differential equation returns
$Be^t\,[3+t + 2+t + 3+3t - 5t] = e^t \iff 8B = 1.$
$\Longrightarrow u_p(t) = \frac{te^t}{8}$ is a particular solution. Thus the general solution is
$u(t) = c_1e^t + c_2\,e^{-t}\cdot\cos(2t) + c_3\,e^{-t}\cdot\sin(2t) + \frac{te^t}{8}$.

Dates of classes: 11-15.12.2023