Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 4, Homework

Exercise 1:

a) Which of the following differential equations for $u(t)$ are exact?

(i) $u + u' = 0$.

(ii) $(12tu+3) + 6t^2 \cdot u' = 0$.

(iii) $2t(u^2 - t^2 - 1) + 2uu' = 0$.

(iv) $u^3 + e^t + 3tu^2u' = 0$.

b) For the exact differential equations in Part a) determine a corresponding potential and the general solution.

Solution:

a) (i) $u + u' = 0$. With $f(t,u) = u$ and $g(t,u) = 1$ it follows $f_u = 1 \neq 0 = g_t$.
Thus the differential equation is not exact.

(ii) $(12tu+3) + 6t^2 \cdot u' = 0$. Here it holds: $f(t,u) = 12tu+3$, $g(t,u) = 6t^2$,
and from this $f_u = 12t = g_t \implies$ The differential equation is exact.

(iii) $2t(u^2 - t^2 - 1) + 2uu' = 0$. With $f(t,u) = 2t(u^2 - t^2 - 1)$ and $g(t,u) = 2u$ it follows $f_u = 4tu \not\equiv 0 = g_t$.
The differential equation is not exact.

(iv) $u^3 + e^t + 3tu^2u' = 0$. With $f(t,u) = u^3 + e^t$ and $g(t,u) = 3tu^2$ it follows $f_u = 3u^2 = g_t$.
Thus the differential equation is exact.

b) We determine a potential Ψ for the differential equation in Part a)ii).
$(12tu+3) + 6t^2 \cdot u' = 0$.

$f(t,u) = 12tu+3,\ g(t,u) = 6t^2$,

$\Psi_t(t,u) = 12tu + 3 \implies \Psi(t,u) = 6t^2u + 3t + c(u) \Rightarrow$

$\Psi_u(t,u) = 6t^2 + 0 + c'(u) \overset{!}{=} g(t,u) = 6t^2$

$\Longleftarrow c'(u) = 0 \iff c(u) = k \iff \Psi(t,u) = 6t^2u + 3t + k$

Solutions of the differential equation satisfy:
$\Psi(t,u) = 6t^2u + 3t + k = \tilde{K} \iff 6t^2u + 3t = K$.

General solution: $u(t) = \dfrac{K - 3t}{6t^2}$ for $t \neq 0$.

A potential for the differential equation from a)iv) is computed as follows:

$$\begin{aligned} \Psi_t &\overset{!}{=} u^3 + e^t && \Rightarrow \quad \Psi = tu^3 + e^t + c(u) \Rightarrow \\ \Psi_u &= 3tu^2 + c'(u) \overset{!}{=} 3tu^2 && \Rightarrow \quad c'(u) = 0 \Rightarrow \Psi(t,u) = tu^3 + e^t + c \end{aligned}$$

$\Rightarrow$ The solutions of the differential equation satisfy
$\Psi(t,u) = tu^3 + e^t = C \ \Rightarrow$

$$u^3 = \frac{C - e^t}{t}$$

General solution: $u = \left(\dfrac{C - e^t}{t} \right)^{\frac{1}{3}}$.

Exercise 2:

a) Determine the solution to the initial value problem

$$u''(t) + 2t^3 u'(t) = e^{-\frac{t^4}{2}} \cdot \sin(2t) \qquad u(0) = 2,\ u'(0) = 0.$$

Hint: It is sufficient to specify an integral representation of the solution.

b) Solve the initial value problem

$$u''(t) = (u(t))^{-3} = g(u(t)), \qquad u(0) = 2,\ u'(0) = 0.$$

Solution:

a) The substitution $y(t) := u'(t)$ returns the following linear differential equation for y.

$$y'(t) + 2t^3 y(t) = e^{-\frac{t^4}{2}} \cdot \sin(2t).$$

Solution formula from the lecture with $a(t) = -2t^3$, $b(t) = e^{-\frac{t^4}{2}} \cdot \sin(2t)$

$A'(t) = -2t^3$ for example $A(t) = -\frac{t^4}{2}$

$e^{-A(t)} \cdot b(t) = e^{\frac{t^4}{2}} \cdot e^{-\frac{t^4}{2}} \cdot \sin(2t) = \cdot \sin(2t) = (B^*(t))'$.

So, for example $B^*(t) = -\frac{1}{2}\cos(2t)$.

$y(t) = e^{A(t)}(B^*(t) + C) = e^{-\frac{t^4}{2}}\left(-\frac{1}{2}\cos(2t) + C\right) = u'(t)$.

The initial value for u' yields

$u'(0) = y(0) = \left(C - \frac{1}{2}\cos(0)\right) e^0 \stackrel{!}{=} 0 \Longrightarrow C = \frac{1}{2}$.

For the solution $u(t)$ we thus obtain

$$u(t) = u(0) + \int_0^t \left(\frac{1}{2} - \frac{1}{2}\cos(2\tau)\right) e^{-\frac{\tau^4}{2}}\, d\tau = 2 + \frac{1}{2}\int_0^t (1 - \cos(2\tau))\, e^{-\frac{\tau^4}{2}}\, d\tau.$$

b) $u''(t) = (u(t))^{-3} = g(u(t))$.

For $u' \neq 0$ the differential equation is equivalent to the equation

$u'u'' = u'u^{-3}$.

From this one gets $\displaystyle\int u'u''dt = \int u'u^{-3}dt \Longleftrightarrow \frac{1}{2}(u')^2 = \int \frac{du}{u^3} = -\frac{1}{2u^2} + C$

Therefore: $\qquad u' = \pm\sqrt{2C - \dfrac{1}{u^2}}$.

For $t = 0$ we know $u(0) = 2$, $u'(0) = 0$. Using this in the formula for u' we get

$$u'(0) = 0 = \pm\sqrt{2C - \frac{1}{u(0)^2}} = \pm\sqrt{2C - \frac{1}{4}} \Longrightarrow C = \frac{1}{8} \Longrightarrow u' = \pm\sqrt{\frac{1}{4} - \frac{1}{u^2}}.$$

Separation or variables yields:

$$\frac{du}{dt} = \pm\sqrt{\frac{u^2 - 4}{4u^2}} = \pm\frac{\sqrt{u^2 - 4}}{2u}$$

$$\int \frac{2udu}{\sqrt{u^2 - 4}} = \pm\int dt \Longleftrightarrow 2\sqrt{u^2 - 4} = \pm t + \tilde{C} \qquad \text{(substitution } z = u^2 - 4\text{)}$$

From the initial condition $u(0) = 2$ we obtain

$$2\sqrt{2^2 - 4} = \pm 0 + \tilde{C} \Longrightarrow \tilde{C} = 0 \Longrightarrow \sqrt{u^2 - 4} = \pm\tfrac{t}{2} \Longrightarrow u^2 - 4 = \tfrac{t^2}{4} \Longrightarrow u(t) = \pm\sqrt{4 + \tfrac{t^2}{4}}.$$

Since $u(0) = 2$, only the positive sign is allowed.

Exercise 3:

The speed at which a solid substance dissolves in a solvent is proportional to the still undissolved quantity of the substance $S(t)$ at time t and to the difference between the saturation concentration and the actual concentration of the already dissolved substance. Let

$$V := \text{ volume} \qquad K_M := \text{ saturation concentration,}$$
$$K_0 := \text{ initial concentration} \qquad S(t) := \text{ undissolved quantity of the substance } S \text{ at time } t,$$
$$S_0 := S(0) = \text{ undissolved quantity of the substance } S \text{ at time zero (initial value),}$$
$$K_0 + \frac{S_0 - S(t)}{V} = \text{concentration of } S \text{ at time } t,$$
$$\gamma := \text{proportionality constant.}$$

a) Describe the diffusion process through a differential equation for $S(t)$.

b) Determine the solution of the initial value problem with data
$S_0 = 10$ kg, $V = 100$ lit, $K_M = 0.25\,\text{kg/lit}$, $K_0 = 0\,\text{kg/lit}$, $\gamma = 4\,\text{lit}/(\text{kg}\cdot\text{s})$.
Use the substitution known from the lecture for logistic growth $u = S^{-1}$.

Solution:)

a) According to the problem, the following applies to the quantity resolved in the time interval Δt:

$$S(t) - S(t+\Delta t) = \gamma \cdot S(t)\left(K_M - K_0 - \frac{S_0 - S(t)}{V}\right)\Delta t,$$

where γ is constant. Thus with $C := K_M - K_0 - \frac{S_0}{V}$

$$S(t) - S(t+\Delta t) = \gamma \cdot S(t)\left(C + \frac{S(t)}{V}\right)\Delta t.$$

The process can be described by the differential equation

$$S'(t) = -\gamma \cdot S(t)\left(C + \frac{S(t)}{V}\right).$$

b) With the given values one has $C := K_M - K_0 - \dfrac{S_0}{V} = 0.25 - 0 - \dfrac{10}{100}$

$$S'(t) = -4S(t)\left(0.15 + \frac{S(t)}{100}\right) = -\frac{1}{25}S(t)(15 + S(t)) = -\frac{3}{5}S(t) - \frac{1}{25}S^2(t).$$

As in the lecture, with $y(t) = S^{-1}(t)$ we get

$$y'(t) = \frac{3}{5}y(t) + \frac{1}{25}$$

Solution of the homogeneous problem:

$$y_h' = \frac{3}{5}y_h \implies y_h = Ce^{\frac{3}{5}t}.$$

Since the coefficients and the inhomogeneity term are constant, one can take the ansatz $y_p = \hat{c} \in \mathbb{R}$ and obtain $y_p = -\frac{1}{15}$.

Alternatively: Variation of constants or solution formula from lecture 1.

Altogether thus:

$$y(t) = Ce^{\frac{3}{5}t} - \frac{1}{15} \implies S(t) = y(t)^{-1} = \frac{15}{ce^{\frac{3}{5}t} - 1}$$

From the initial value $S(0) = 10$ finally one finds $c = \frac{5}{2}$ and from this the uniques solution

$$S(t) = \frac{30}{5e^{\frac{3}{5}t} - 2}$$

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