Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
Prof. Dr. T. Schmidt, Dr. H. P. Kiani E. Ficola

Differential Equations I for Students of Engineering Sciences

Sheet 4, Exercise class

Exercise 1)

Determine which of the following differential equations is separable, linear, Bernoulli, Riccati or a similarity differential equation. If necessary provide a substitution which transforms the differential equation into a separable or linear differential equation. How do the new differential equations obtained by substitution read?

Hint: you don't have to solve the differential equations, but feel free to do so!

a) $(1+e^{2t})u' = -2e^{2t}u$

b) $u' - 2t^2(u-1) + tu(u-2) = 1 - t - t^3$. Hint: there is a solution $u_p(t) = \alpha t + \beta$.

c) $\cos(t)u' + \sin(t)u = -\cos^2(t)u$

d) $u - \frac{1}{t} - \frac{1}{u}u' = 0$

e) $u' = 2t(2t^2u^2 - 1)u$

f) $u - tu' = \dfrac{t^3}{u^2}$

Solution:

a) $u' = \frac{-2e^{2t}}{1+e^{2t}}u$ is separable and it can be directly solved.

b)
$$u' - 2t^2(u-1) + tu(u-2) = 1 - t - t^3 \iff u' - (2t^2+2t)u + tu^2 = 1 - t - 2t^2 - t^3.$$

The differential equation is Riccati with $a(t) = 2t^2 + 2t$ and $b(t) = -t$.
The ansatz $u_p = \alpha t + \beta$ inserted into the differential equation yields:

$$\alpha - (2t^2+2t)(\alpha t + \beta) + t(\alpha t + \beta)^2 = 1 - t - 2t^2 - t^3.$$

Comparison of coefficients returns
$\alpha = \beta = 1$ also $u_p(t) = t + 1$.
Now we set $y = \frac{1}{u-u_p}$ and according to the lecture/auditorium exercise class obtain the equation for y:

$$y' = -[a(t) + 2b(t)u_p(t)]y - b(t) \iff y' = -[2t^2 + 2t - 2t(t+1)]y = t$$

Thus $y'(t) = t$. This is not only a linear differential equation , but even a directly integrable equation.

c) $\cos(t)u' + \sin(t)u = -\cos^2(t)u \iff u' = [-\tan(t) - \cos(t)]\,u.$

The equation is separable.

d) $u - \frac{1}{t} - \frac{1}{u}u' = 0 \iff u^2 - \frac{u}{t} - u' = 0 \iff u' = -\frac{1}{t}u + u^2.$

The differential equation is Bernoulli with

$$\alpha = 2,\ a(t) = -\frac{1}{t},\ b(t) = 1.$$

One substitutes $y = u^{-1}$ and obtains the new differential equation :

$$y' = -1 \cdot (-\frac{1}{t}y + 1) = \frac{1}{t}y - 1.$$

e) $u' = 2t(2t^2u^2 - 1)u = 4t^3u^3 - 2tu \iff u' = -2tu + 4t^3u^3$
is Bernoulli with

$$\alpha = 3,\ a(t) = -2t,\ b(t) = 4t^3.$$

One substitutes $y = u^{-2}$ and obtains the linear differential equation :

$$y' = -2 \cdot (-2ty + 4t^3) = 4ty - 8t^3.$$

f) $u' = \frac{u}{t} - \left(\frac{t}{u}\right)^2$ is a similarity differential equation. The substitution $y = \frac{u}{t}$ leads to the differential equation

$$y' = \frac{-y^{-2}}{t}.$$

This is separable.

Exercise 2)

Determine the general solution of the differential equation

$$u'(t) + 2u(t) - t\,u(t)^4 = 0.$$

Solution 2:

The differential equation is a Bernoulli equation.

With $\alpha = 4$, $a(t) = -2$ and $b(t) = t$ and $y = u^{1-\alpha} = u^{-3}$ one obtains the linear differential equation in y

$$y'(t) = (1-\alpha)(a(t)y(t) + b(t)) = -3(-2y(t) + t) = 6y(t) - 3t.$$

Version 1) Variation of constants.

$y_h' = 6y_h \implies y_h(t) = e^{\int 6dt} = Ce^{6t}$.

Variation of constants

$y_p(t) = C(t)e^{6t} \overset{\text{ODE}}{\longrightarrow} C'(t)e^{6t} \overset{!}{=} -3t$

$$C(t) = \int -3te^{-6t}dt = \left[-3t\frac{e^{-6t}}{-6}\right] - \int -3\frac{e^{-6t}}{-6}dt = \frac{t}{2}e^{-6t} - \frac{1}{2}\int e^{-6t}dt$$

$$= \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t} + K.$$

Thus for example with $K = 0$

$y_p(t) = C(t)e^{6t} = \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t} \cdot e^{6t} \implies y_p(t) = \frac{t}{2} + \frac{1}{12}$.

$y(t) = y_h(t) + y_p(t) = Ce^{6t} + \frac{t}{2} + \frac{1}{12}$.

Version 2) Solution formula from Lecture 1.

$A'(t) = \hat{a}(t) = 6$ for example $A(t) = 6t$.

With $\hat{b}(t) = -3t$ we compute

$e^{-A(t)} \cdot \hat{b}(t) = e^{-6t} \cdot (-3t) = -3te^{-6t} = (B^*(t))'$.

$$B^*(t) = \int -3te^{-6t}dt = \left[-3t\frac{e^{-6t}}{-6}\right] - \int -3\frac{e^{-6t}}{-6}dt = \frac{t}{2}e^{-6t} - \frac{1}{2}\int e^{-6t}dt$$

$$= \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t} + K.$$

The choice $B^*(t) = \left(\frac{t}{2} + \frac{1}{12}\right)e^{-6t}$ returns

$y(t) = e^{6t}(B^*(t) + C) = e^{6t}\left((\frac{t}{2} + \frac{1}{12})e^{-6t} + C\right)$.

Since $y = u^{1-\alpha} = u^{-3}$ we obtain

$u(t) = \frac{1}{\sqrt[3]{y}}$.

Exercise 3: Compute the solution of the differential equation

$$u'(t) = 1 - t + t^2 + u(t) - 2tu(t) + (u(t))^2.$$

Hint: there exists a polynomial solution $u_p(t) = mt + k$.

Solution: The differential equation is Riccati with $a(t) = 1 - 2t,\ b(t) = 1, c(t) = 1 - t + t^2$.

Inserting the ansatz $u_p = mt + k$ in the differential equation returns

$$m \overset{!}{=} 1 - t + t^2 + mt + k - 2mt^2 - 2kt + m^2t^2 + 2mkt + k^2$$

Thus

$0 \cdot t^2 + 0 \cdot t + m \overset{!}{=} (1 - 2m + m^2) \cdot t^2 + (-1 + m - 2k + 2mk) \cdot t + (1 + k + k^2)$

Comparison of the coefficients yields:

$0 = (1 - 2m + m^2) = (1 - m)^2 \rightarrow m = 1$,

$0 = (-1 + m + 2mk - 2k) \rightarrow 0 = (-1 + 1 + 2k - 2k) = 0$,

and

$m = (1 + k + k^2) \rightarrow 0 = k(k + 1)$.

For example we may choose $k = 0$ and from this $u_p = t$.

Substitution: $y := \frac{1}{u-u_p} = \frac{1}{u-t}$

Thus from the lecture we obtain:

$$y' = -[a(t) + 2u_p(t)b(t)]y(t) - b(t) = -[1 - 2t + 2t]y(t) - 1 = -y(t) - 1$$

The solution of the homogeneous differential equation $y' = -y$ is now well known: $y_h(t) = ce^{-t}$.

A particular solution $y_p(t) = -1$ of the inhomogeneous differential equation can be guessed through the ansatz $y_p(t) = \hat{c}$. Compute with the formula from the first lecture or by variation of the constants:

$y_p(t) := c(t)e^{-t} \overset{\text{ODE}}{\Longrightarrow} c'(t)e^{-t} = -1 \Longrightarrow c'(t) = -e^t \Longrightarrow c(t) = -e^t + C$

Thus for example (with $C = 0$): $y_p(t) = -e^t e^{-t} = -1.$

And from this $y(t) = ce^{-t} - 1$.

Reverse transformation:

$y = \frac{1}{u-t} \Longrightarrow u - t = \frac{1}{y} = \frac{1}{ce^{-t}-1} \Longrightarrow u(t) = t + \frac{1}{ce^{-t}-1}.$

Dates of classes: 27.11-01.12.2023.