Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 3, Homework

Exercise 1:

a) Determine the solution of the following initial value problem

$$y'(t) = t + t(y(t))^2 \qquad \text{for } t > 0, \qquad y(0) = 1.$$

On which interval $I = [0, t^*)$ is its solution defined?

b) Find the general solution of the following differential equation.

$$y'(t) = e^{-2t} \cdot \sqrt[3]{y(t)}.$$

c) Which solution results in b) if the initial value $y(0) = 1$ is given?

d) Which solution results in b) if the initial value $y(0) = 0$ is given?

Sketch of solution of Exercise 1)

a) It is a separable differential equation.One computes
$\frac{dy}{dt} = t(1+y^2) \iff \frac{dy}{1+y^2} = tdt.$ **(1 point)**
Thus
$\int \frac{1}{1+y^2} dy = \int tdt \iff \arctan(y(t)) = \frac{t^2}{2} + C.$
$\iff y(t) = \tan\left(\frac{t^2}{2} + C\right).$ **(1 point)**

$y(0) = \tan\left(\frac{0^2}{2} + C\right) \overset{!}{=} 1 \iff C = \frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}.$ **(1 point)**

As $\tan(\alpha + k\pi) = \tan(\alpha), \forall k \in \mathbb{Z}, \forall \alpha \in \mathbb{R}$, one may choose any k to represent the function. Thus for example $y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right)$. **(1 point)**

The solution is only defined for $\frac{t^2}{2} < \frac{\pi}{4}$. The tangent function has poles for the arguments $k\pi + \frac{\pi}{2},\ k \in \mathbb{Z}.$ **(1 point)**

b) It is another separable differential equation.
$\frac{dy}{dt} = e^{-2t} \cdot \sqrt[3]{y(t)} \overset{(*)}{\iff} \frac{dy}{\sqrt[3]{y}} = e^{-2t}dt.$ **(1 point)**
From this one has
$\int y^{-\frac{1}{3}} dy = \int e^{-2t} dt \iff \frac{3}{2} y^{\frac{2}{3}} = \frac{e^{-2t}}{-2} + \tilde{C}$ **(1 point)**

$\iff \sqrt[3]{y^2(t)} = C - \frac{e^{-2t}}{3} \qquad (C = \frac{2}{3}\tilde{C})$
Thus $y(t) = \pm\sqrt{\left(C - \frac{e^{-2t}}{3}\right)^3}.$ **(1 point)**

c) $y(0) = \pm\sqrt{\left(C - \frac{e^0}{3}\right)^3} = \pm\sqrt{\left(C - \frac{e^0}{3}\right)^3} \overset{!}{=} 1$
Only the positive sign is allowed and it returns $C = \frac{4}{3}$. **(1 point)**

d) $y(0) = \pm\sqrt{\left(C - \frac{e^0}{3}\right)^3} = \pm\sqrt{\left(C - \frac{e^0}{3}\right)^3} \overset{!}{=} 0$
It yields $C = \frac{1}{3}$ but there is no indication on the sign. **(1 point)**
We will come back to this problem later on. Here one may already notice that the first transformation $(*)$ in Part b) can only be carried out for $y \neq 0$.

Exercise 2:

a) Consider the differential equation

$$y'(t) \;=\; f(\alpha t + \beta y(t) + \gamma)$$

with $\alpha,\, \beta,\, \gamma \in \mathbb{R}$ and $\alpha + \beta f(\alpha t + \beta y(t) + \gamma) \neq 0$.
Show that with the help of the substitution

$$u(t) := \alpha t + \beta y(t) + \gamma$$

it can be transformed into a separable differential.

b) Determine the general solution of the differential equation

$$y' \;=\; 1 + \frac{2}{t - y + 4}, \qquad \text{for } t - y + 4 > 0\,.$$

c) Check the solution obtained in Part b) by inserting it into the differential equation.

Sketch of solution for Exercise 2:

a)

$$u(t) := \alpha t + \beta y(t) + \gamma \Longrightarrow u' = \alpha + \beta y' \;=\; \alpha + \beta f(u)\,.$$

$$\frac{du}{\alpha + \beta f(u)} \;=\; dt\,. \qquad \textbf{(2 points)}$$

b) From Part a) it follows with $\alpha = 1,\, \beta = -1,\, \gamma = 4$ and $f(u) = 1 + \dfrac{2}{u}$

$$u' \;=\; 1 - \left(1 + \frac{2}{u}\right) = -\frac{2}{u}\,. \qquad \textbf{(2 points)}$$

Thus we obtain

$$\int u\, du \;=\; -\int 2dt \Longrightarrow \frac{u^2}{2} \;=\; -2t + \tilde{c} \Longrightarrow u(t) = \pm\sqrt{c - 4t}\,. \qquad \textbf{(2 points)}$$

The solution is only defined for $c - 4t > 0$. Transforming back yields

$$y(t) = t + 4 \mp \sqrt{c - 4t}\,. \qquad \textbf{(1 point)}$$

c) Inserting into the differential equation on the left-hand side returns:

$$y' = 1 \mp \frac{1}{2\sqrt{c - 4t}} \cdot (-4) = 1 \pm \frac{2}{\sqrt{c - 4t}}$$

and on the right-hand side

$$1 + \frac{2}{t - y + 4} = 1 + \frac{2}{t - (t + 4 \mp \sqrt{c - 4t}) + 4} = 1 \pm \frac{2}{\sqrt{c - 4t}}\,. \qquad \textbf{(3 points)}$$

Hand in until: 17.11.2023